COURSE GUIDEBOOK

The Joy of Mathematics

Part II

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Course No. 156

The Joy of Mathematics, Part II
Professor [illegible] H. Siegel

COURSE GUIDEBOOK

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The Joy of Mathematics

Part II

Professor Murray H. Siegel
Sam Houston State University

THE TEACHING COMPANY®

Murray H. Siegel, Ph.D.

Sam Houston State University

Although Dr. Siegel is known nationally as a mathematics leader in our public schools, much of his professional life has been devoted to adult education. His community workshops, college courses, college workshops, and videos have one purpose. The primary objective is to help adults overcome mathematical anxiety and to provide his audiences with a picture of mathematics as a subject with logical underpinnings and great utility. In addition, he tries to focus on the connectivity of the various branches of mathematics, as well as the beauty that exists throughout the subject.

Murray Siegel was born and raised in Brooklyn, NY. He was educated in New York City schools, where he met two of his three "Great Teachers." Mary Doyle (his sixth-grade teacher) and Sally Woroner (his ninth-grade teacher) inspired him with the knowledge that he had no limits other than those he placed on himself and that there was no excuse for accepting anything less than excellence. He received a B.S. in Physics from New York University College of Engineering. At his alma mater he met his third "Great Teacher," Harry Park, who allowed him to look more clearly at himself so that he could know others better. Dr. Siegel accomplished his graduate studies in mathematics education at Georgia State University, where he received his M.Ed. Ed.S. and Ph.D. Dr. Siegel is currently Assistant Professor of Mathematics in the Department of Mathematical and Information Sciences at Sam Houston State University, Huntsville, TX.

After a stint in the USAF and in business, Dr. Siegel decided that he really belonged in teaching. Of course, his engineering, military, and business experience has allowed him to see mathematics as a tool. He has attempted to pass this view along to all his students, adults as well as children. Dr. Siegel claims that teaching mathematics is a missionary vocation. This tape series is truly an attempt to bring the meaning of mathematics to many.

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The Joy of Mathematics
Part II

The Joy of Mathematics

Scope:

This twenty-four lecture series has seven general topics, plus a concluding lecture that should bring closure to your experience. The general topics are: numbers, algebra, calculus, fractals, geometry, probability, and data analysis.

Part I

Numbers

The first five lectures of the series attempt to trace the development of the use of numbers by humans from the Egyptian hieroglyphics to our current number system. The various types of numbers we use, including whole numbers, integers, rational numbers, irrational numbers, and complex numbers, are investigated. The sense that the rules of numbers are arbitrary or "magical" should be replaced with a sense of understanding about *why* we do what we do and *who* did it first.

Algebra

Lectures Six through Nine are devoted to the demystification of algebra. Algebra is shown to be a useful tool in understanding arithmetic. A significant amount of time is spent investigating the use of algebraic functions to model situations in our world. A linear function is used to explore the relationship between fat and calories in pizza. A quadratic equation models the trend in the population density of the United States. A cubic function provides a visual image of the growth of the number of high school soccer players. The acceleration of federal expenditures on social insurance is diagnosed with an exponential function. The utility of algebra is exposed so that no student should ever ask, "When am I going to use algebra?"

Calculus

Two lectures provide a brief acquaintance with calculus. The seventeenth century was a wonderful time for new ideas. Kepler had produced mathematical models for planetary motion but there was no mathematics available to deal with a physical measure that had velocity and acceleration. Working independently, Newton and Leibniz produced calculus. Some years later, Taylor and Maclaurin used calculus to create power series. Lectures Ten and Eleven look at the derivative and the integral, followed by a view of power series seen through the use of a graphing calculator.

Fractals

Lecture Twelve is devoted to the beauty of fractals. Investigations into this branch of mathematics started at the end of the nineteenth century. Lack of computer power kept the investigations restricted to the abstract. In the last part of the twentieth century, we have been awed by the complexity of the fractal

image. But fractals are not simply art; they are being used to allow scientists to understand chaotic phenomena such as hurricanes. Part I of the course ends with Lecture Twelve.

Part II

Geometry

We start Part II of our exploration into the joys of math by going back into history and attempting to appreciate the great geometric thinkers of the ancient world (Lecture Thirteen). Thales, Pythagoras, Euclid, and Apollonius built a foundation of understanding that we use today. The patterns of geometric relationships are discovered and the rationale for all those geometric formulas is brought to light. René Descartes took his love of geometry and applied it to algebraic problems and created analytic geometry. The graph paper we use to provide models for the abstract was his invention. Did you ever wonder where they obtained the names for the trigonometric functions—sine, tangent, and secant? Actually, one of the names was created by a mistaken translation, but the other names make a great deal of sense. Lecture Fifteen will discuss triangles and trigonometry, and Lecture Sixteen covers conic sections.

Probability

We live in an uncertain world. Insurance is based on uncertainty and many people view investment as a form of gambling. Probability is the mathematical analysis of this phenomenon. It turns out that gambling played a significant role in the early development of the mathematics of probability. This section will investigate the binomial and normal distributions and how we can apply them in problem-solving situations. The use of simulations to provide data when none are available is demonstrated. After viewing these three lectures on probability (Lectures Seventeen through Nineteen), you should have a better understanding of the meaning of "the chance of rain tomorrow is thirty percent."

Data Analysis

Four lectures involve the treatment of data. Analyzing a set of data, comparing sets of data, and seeking a statistical model for a relationship between two variables are discussed. The use of sample results to make an inference about the population is the focus of the last two lectures on data analysis. The use of the hypothesis test to determine a level of significance and the use of a confidence interval to estimate a population parameter are both explained using realistic examples. How do they know how many people support a particular political philosophy?

The final lecture starts with the question, "What would mathematics be if our ancestors had only had eight fingers rather than ten?" A new perception of mathematics is the primary objective of this series. By Lecture Twenty-Four, the walk through history, the breaking down of mathematical secrets, the use of

pattern recognition, and the exposure to the beauty of mathematics should allow the viewer to have that new (and hopefully more positive) view of the subject.

Note: Dr. Siegel has made two other mathematics courses for The Teaching Company. These would be appropriate for students who would like a review of basic mathematical concepts and operations. The courses are *Basic Math* and *Algebra II*.

Additionally, The Teaching Company has two other high-school math courses: *Geometry* by James Noggle and *Algebra I* by Dr. Monica Neagoy, both of which cover topics discussed in *The Joy of Mathematics*.

Lecture Thirteen

Geometry in History

Scope: Other than counting, the first mathematical thinking that humans did involved the mathematics of shape. This lecture reviews the development of geometry from the first Egyptian pyramid to the advent of analytic geometry in the seventeenth century. Geometry began as a practical art, but the study of geometry led to the development of logic and the desire to establish truths. Geometry was the concrete image for mathematical work. Once analytic geometry was created, geometry took a back seat to algebra as the vehicle used to make progress in mathematics.

Outline

I. Geometry began with a practical need to measure shapes.

A. The word "geometry" means to measure the earth.

1. It is said that geometry became important when the Egyptian pharaoh wanted to tax the farmers who raised crops along the Nile.
2. That pharaoh was Sisostris. He ruled that the land had been given to the farmers by the gods and the gods caused the Nile to overflow, bringing rich soil to the farmers' lands.
3. To compute the correct amount of tax, the government agents had to be able to measure the amount of land being cultivated.
4. Humans had certain understandings about shape, even before recorded history. The Egyptians made a science out of the study of shape.

B. As civilization developed, geometry became more complex.

1. Around 2900 B.C., the first Egyptian pyramid was constructed. Knowledge of geometry was vital to those who directed this construction.
2. The earliest record of a formula for calculating the area of a triangle indicates that the Babylonians accomplished this around 2000 B.C.
3. Knowledge of angle measures is critical if a sundial is to provide accurate information. There is evidence that Egyptians had sundials by 1500 B.C.
4. It is believed that knowledge of geometric principles was developing in ancient India and China, but no records exist to prove this. Chinese and Indian scholars wrote on materials that have decayed over the centuries. The Egyptians and Babylonians left many of their records on clay.

II. In the Greek world, there were three great philosophers whose works in geometry are still studied today.

A. Thales of Miletus was the first great geometer.

1. Thales lived in Egypt and Greek-controlled Asia Minor during the sixth century B.C.
2. He was also considered a statesman, an engineer, a businessman, and an astronomer.
3. His ability as a problem solver was demonstrated when King Croesus of Lydia needed a bridge constructed to allow his army to cross a river in pursuit of an enemy force.
4. The king was in a hurry and called upon Thales to find a solution. Thales had the soldiers dig a channel to divert the river's waters, which allowed the army to walk across on the riverbed.
5. It is said that after the army had crossed the river, Thales demanded that the channel be refilled with soil to return the river to its original path.
6. Thales studied similar triangles and wrote the proof that corresponding sides of similar triangles are in proportion.
7. It was written that Thales's knowledge of astronomy caused him to predict a solar eclipse that is believed by modern scientists to have occurred on May 28, 585 B.C.
8. There is great significance to that date. A battle between armies from Lydia and Media was halted because of the soldiers' fears that the eclipse was a terrible omen.
9. Because we know the exact date of the eclipse, this battle is the earliest historical event for which we have an exact date.
10. The most significant contribution for which Thales is given credit is the development of deductive procedures.

B. The next great Greek geometer was Pythagoras, who also lived in the sixth century B.C.

1. It is thought that Pythagoras studied under Thales. Unfortunately, little is known about Pythagoras. There are some scholars who doubt that he existed.
2. Pythagoras founded a brotherhood, the Pythagoreans, who pursued knowledge in mathematics, science, and philosophy.
3. The Pythagoreans developed properties of parallel lines, proportions, and similar figures.
4. The Pythagoreans produced a musical scale by relating the length of a string on a lyre to the pitch produced by plucking the string. Experimentation with bells and containers of water verified the proportional relationship.
5. The most famous contribution of the Pythagoreans was the Pythagorean theorem: The sum of the squares of the legs of a right triangle equals the square of the hypotenuse.

6. We know that the builders of the Egyptian pyramids knew about the 3–4–5 right triangle. We know that the Chinese understood this relationship, perhaps as early as 1000 B.C., but it is Pythagoras who gets the credit.
7. There have been a number of different proofs of the Pythagorean theorem. Perhaps the most unexpected source of a proof was James Garfield. Garfield used a trapezoid to prove the theorem. This proof was completed many years before Garfield was elected president.

C. Euclid is considered the "father of geometry."
1. Euclid was a Greek mathematician who lived around 300 B.C.
2. He wrote *The Elements*, a text that covered all the knowledge about geometry, number theory, and algebra that existed at that time.
3. Euclid's goal was to present an organized work that emphasized deductive reasoning.
4. He established a few basic postulates, the truth of which was accepted. From this beginning, theorems were proven using the postulates and, eventually, previously proven theorems.
5. This deductive process is called the Euclidean process and is used in all branches of mathematics.
6. The geometry that is studied in high school today is Euclidean and would be recognizable to Euclid.
7. Geometries were developed in the nineteenth and twentieth centuries that had different foundations. These geometries are classified as non-Euclidean.

III. Work in geometry was conducted on an isolated basis for hundreds of years after Euclid. Much of what was known in the fourth century B.C. was lost and had to be rediscovered.

A. Work on proofs and the abstract was stifled by Rome. Practical mathematics was required, and there was no need for proving theorems.
1. The knowledge discovered and developed by the Greeks was forgotten in Europe.
2. The great Greek works were kept in the Library of Alexandria, and Europeans were isolated from the knowledge contained in these texts.
3. Arab armies conquered North Africa and absorbed the learning that had been preserved in Alexandria.
4. Arab armies also invaded India, where they discovered that Indians had been working on the arithmetic of geometry.
5. Brahmagupta, in 628, had studied the sides, diagonals, and areas of various polygons.
6. The wars and trade between Europe and the Muslim world caused the knowledge of the ancient world and the knowledge developed

in North Africa, Muslim Spain, and India to be transported to European centers of scholarship.

B. The rebirth of academic pursuits in Europe ignited a search for geometric knowledge.
1. In 1260, a Latin translation of Euclid's *Elements* became a text at universities in the major European cities.
2. The development of art during the fifteenth century Renaissance required an understanding of geometric perspective. Paintings had to show depth and knowledge of proportion was vital.
3. In 1453, Arab armies conquered the city of Constantinople, the last bastion of Rome in the East. Greeks who fled the city relocated to Italy and bought with them their knowledge of Greek mathematics.
4. The invention of the printing press allowed knowledge of geometry to spread rapidly.

C. René Descartes was dissatisfied with the ancient thought that he had been taught while studying at the University of Poitiers. He sought to create a new philosophy.
1. Descartes believed that the universe was created with a divine mathematical plan.
2. His philosophy maintained that the laws of nature could not be varied. He believed that all geometric relationships could be described mathematically.
3. In 1637, Descartes published *Discourse on the Method*, in which he introduced analytic geometry.
4. He was the first to align two number lines in a perpendicular fashion to create a graphing plane. Today that plane is called the Cartesian plane in his honor.
5. Descartes studied geometric properties by using coordinates in his plane and by studying equations that produced the various shapes.
6. His work *Principia Philosophiae* was published in 1649, a year before his death. It included his thoughts on mathematics, science, and astronomy.

D. Returning to Pythagoras, let's investigate a sequence of numbers called "Pythagorean triples."
1. The numbers 3–4–5 are one example; 5–12–13 are another; and 7–24–25 are yet another.
2. Can you determine the rule for the sequence? Here's the answer: The square of the first number equals the sum of the second two numbers.

Essential Reading:

NCTM, *Historical Topics for the Mathematics Classroom*, Section IV.

Sanderson Smith, *Agnesi to Zeno*.

Supplementary Reading:

E. T. Bell, *Men of Mathematics*.

William Dunham, *The Mathematical Universe*.

Morris Kline, *Mathematical Thought From Ancient to Modern Times*, Volume 1.

James R. Newman, *The World of Mathematics*, Volume One.

Lynn Arthur Steen, *On the Shoulders of Giants*, DIMENSION and QUANTITY.

Questions to Consider:

1. Compare European paintings before 1300 A.D. and after 1400 A.D. in terms of perspective. An excellent source of difference is the treatment of human feet, shod or otherwise.
2. Research the "parallel postulate." It is a fundamental postulate in Euclid's *Elements* that has been shown to be untrue. Find out how the failure of the parallel postulate led to the creation of non-Euclidean geometries.

Lecture Fourteen

Geometric Relations: Area, Volume, and Polygons

Scope: This lecture is devoted to important relationships in geometry and the history of their development. Area, we have seen, was the rationale for the creation of geometry as a science. This lecture investigates the area formulas for a number of common shapes. The development of formulas for volume was slower than for area. One of the great moments in this development was caused by a concern about the cost of a cask of wine. Calculus aided in the discovery of formulas for the volume of various solids. The relationships for the measure of the angles of a regular polygon and the number of diagonals in a polygon are investigated. At the end of this lecture, triangular, rectangular, and square numbers are discussed, which leads to a look at the geometric representation of arithmetic.

Outline

I. The previous lecture mentioned that the need to compute the area of tracts of farmland was the reason for the genesis of geometry as a science.

A. Much time was spent developing formulas for the area of various two-dimensional shapes.

1. One of the earliest recorded geometry problems is a challenge to find a square that has an area that is equal to the area of a given circle.
2. This problem is even mentioned in the play *The Birds* written by Aristophanes in 414 B.C.
3. Because of its unique shape, finding the circle's area was a goal of many philosophers. It was understood that the ratio between the circumference and the diameter of a circle was a key piece of knowledge.
4. As we have seen in the lecture on the irrational numbers, the ancients believed that the ratio could be written as a fraction.
5. In 1800 B.C., the Babylonians used 3 1/8 for this ratio, while in 1650 B.C., the Egyptians used 3.16.
6. In the third century B.C., Archimedes used a polygon that was inscribed in a circle, which was inscribed in another polygon, to estimate that the ratio was between 223/71 and 22/7.
7. Archimedes wrote that the area of a circle was equal to that ratio multiplied by the square of the radius.
8. In 480 A.D., Zu Chongzhi, of China, defined the ratio that we call π as 355/113 or 3.1415929, which is correct to six decimal places. Indian mathematicians Aryabhata (530 A.D.) and Bhaskara

(twelfth century) produced more accurate rational approximations of π.

9. In the seventeenth century, Johannes Kepler wrote the formula for the area of a circle as one-half times the radius times the circumference.

B. The formulas for the area of common polygons started with the square and the formula for one type was used to obtain the formula for the next.

1. The square is the basic unit of area. The number of unit squares one can fit on a surface is the area of that surface.
2. The number of unit squares that can fit in a square with a side that measures 5 is obviously 5 x 5, or 5^2.
3. There is evidence that Egyptians had sundials by 1500 B.C. Knowledge of angle properties must have been known for them to accomplish this.
4. The use of the word "square" for the second power comes from the fact that the formula for the area of a square is the second power of the side. Raising the side to the second power squares the side.
5. A rectangle that has a length of L and a width of W has L x W unit squares in it. Therefore, the formula for the area of a rectangle is L x W.
6. Removing a right triangle from one side of a parallelogram and pasting it to the other side demonstrates that a parallelogram has an area equal to a rectangle whose length and width have the same measures as the base and height of the parallelogram.
7. The area of a parallelogram is the product of base and height. It is interesting to note that the area of a parallelogram does not vary as we vary the base angle, provided the base and height stay the same.
8. Because any triangle when reflected about any side will produce a parallelogram, the area of a triangle must be one-half of the product of its base and height.
9. A trapezoid can be divided into two triangles by drawing a diagonal. Each triangle has the same height. Thus, the area of a trapezoid is one-half the product of the height and the sum of the two bases.

II. Formulas for volume were more difficult to determine. The discovery of calculus accelerated the development of these formulas.

A. The volume of a cube and a rectangular prism were easy to find. Other figures presented a greater challenge.

1. Archimedes used his method of exhaustion to find the volumes of various solids.
2. It was not until 1615 that a serious attempt was made to measure volumes.

3. In that year, Kepler sought to find the volume of a cylinder. His motivation was his concern over the cost of casks of wine that he was purchasing for the celebration of his second marriage.
4. Kepler attempted to measure the volume of thin layers as he went from bottom to top of the cylindrical barrel. He needed the derivative to complete his work but that was not yet available.
5. Using calculus, it is simple to determine that the volume of a cylinder is the product of the area of its base and its height ($V = \pi r^2 h$).
6. Indeed, the formula for the volume of any prism is the product of the area of its base and its height.
7. A prism is a solid with two bases and rectangular faces connecting the bases. The cylinder is simply a prism with circular bases.

B. Once the volume of a prism was known, the formula for the volume of a pyramid was easily obtained.

1. In 1650 B.C., the Egyptians showed that the volume of a square-based pyramid was one-third the volume of a square-based prism with the same height.
2. Calculus allowed the generalization that the volume of any pyramid is one-third the volume of a prism with the same base and the same height.

C. Archimedes did discover that the volume of a sphere is 4/3 times the product of π and the cube of the radius.

III. A polygon is a closed figure with line segments for sides, all of which are attached to adjacent sides at their end points. In a convex polygon, a line segment connecting any two points in the interior of the polygon is itself totally within the interior. A regular polygon has congruent sides and congruent angles.

A. First, we shall establish a relationship between the number of sides and the measure of the interior angles of a regular polygon.

1. The polygon with the smallest number of sides is the triangle. As we increase the number of sides by one, we add one more triangle or 180 degrees to the sum of the interior angles.

2.

Number of Sides	Total of Interior Angles	Each Angle
3	180	60
4	360	90
5	540	108
6	720	120
8	1080	135
12	1800	150
20	3240	162

3. Algebraically, the measure (m) of an interior angle in a regular polygon with n sides is given by $m = 180(n - 2)/n$.

B. Our next task will be to find the sum of the exterior angles for a polygon.

1. An exterior angle is created by extending a side of a polygon. The exterior angle is formed by that extension and the adjacent side that shares the same vertex.
2. The interior and exterior angles form a linear pair since they form a straight line. Thus, the two angles must be supplementary.
3. Using regular polygons:

Number of Angles	Interior Angle	Exterior Angle	Sum of Exterior Angles
3	60	120	360
4	90	90	360
5	108	72	360
6	120	60	360
8	135	45	360
12	150	30	360
20	162	18	360

4. It would appear that the sum of the exterior angles for any polygon is equal to 360 degrees.

C. Our final analysis of polygons involves the number of diagonals and the total number of line segments for a given number of sides.

1.

Number of Sides	Number of Diagonals	Number of Total Line Segments
3	0	3
4	2	6
5	5	10
6	9	15

2. There is a definite pattern to the sequence of the number of line segments. As we add a side, we increase the number of vertices by one. The new vertex must be connected to the previous vertices.
3. When we add a side to a polygon, we add the next counting number to the total number of line segments.
4. The pattern in the number of total line segments that we observe has a name and we will study it in the final part of this lecture.

IV. Figurate numbers, numbers that are associated with a geometric shape, have fascinated people for centuries. We will take a brief look at the triangular numbers, the square numbers, and the rectangular numbers.

A. The triangular numbers form a sequence of numbers in which the numbers in the sequence, if displayed as a number of dots, could form a triangular shape.

1. Six is a triangular number, because we could have three dots on the bottom row, two dots on the second row, and one dot on the top row.
2. By adding four dots, we have the next triangular number, which is ten.
3. The sequence of triangular numbers is 0, 1, 3, 6, 10, 15, 21, 28, 36, 45, 55, 66, 78, etc.
4. The sequence of the number of line segments associated with the polygons is the triangular number sequence.
5. Adding any two adjacent triangular numbers produces a sum that is a square number ($6 + 10 = 16$, $36 + 45 = 81$).
6. Triangular numbers were first identified as a sequence by Nicomachus of Gerasa (Greece) in A.D. 100.

B. The square numbers are numbers obtained by multiplying a whole number by itself. A square number of dots can be arranged in rows to form a square.

1. The sequence of square numbers is 0, 1, 4, 9, 16, 25, 36, 49, 64, …
2. The geometric representation of the square numbers provides a clue to the pattern observed when looking at the sequence—you always add the next odd number to a square number to obtain the next square number ($0 + 1 = 1$, $1 + 3 = 4$, $4 + 5 = 9$, $9 + 7 = 16$).

C. Rectangular numbers are numbers that can be arranged in rows to form a rectangle.

1. All counting numbers are rectangular numbers.
2. A number that has only one rectangular arrangement is a prime number.
3. A number that has a rectangular arrangement that is square must be a square number.
4. The rectangular arrangements provide a visual representation of the various factors of the number.

5. Using this approach allows us to teach even small children something about the relationships between and among the various counting numbers.

Essential Reading:

NCTM, *Historical Topics for the Mathematics Classroom*, Sections II and IV.

Sanderson Smith, *Agnesi to Zeno*.

Supplementary Reading:

E. T. Bell, *Men of Mathematics*.

William Dunham, *The Mathematical Universe*.

Morris Kline, *Mathematical Thought From Ancient to Modern Times*, Volume 1.

James R. Newman, *The World of Mathematics*, Volume One.

Questions to Consider:

1. What is a pentagonal number? Find a relationship between the pentagonal numbers and the triangular numbers.
2. Why could the ancients not find a square that had an area equal to the area of a specified circle?

Lecture Fifteen

Triangles and Trigonometry

Scope: This lecture deals specifically with triangles. The geometry of the triangle will be thoroughly investigated, including a look at similarity and congruence. The second part of the lecture is a discussion of the development and fundamental principles of trigonometry. The last portion is concerned with the applications of trigonometry. Included in this is the use of graphical displays of trigonometric functions to model cyclical situations.

Outline

I. The triangle is one of the building blocks of geometry. We will review information about the importance of triangles, the various parts of a triangle, the classification of triangles, and relationships between triangles.

A. From the earliest study of geometry, the triangle was important. A great deal of time was spent by scholars investigating the triangle.

1. A triangle is a simple shape. It is the polygon with the least number of sides.
2. As was demonstrated in the previous lecture, the triangle is the building block of polygons. Each time we add a side to a polygon, we are adding a triangle to the interior.
3. The triangle is a very stable shape. It is used in all types of construction because of its stability.
4. The Pythagorean theorem has elevated the importance of the right triangle.
5. Triangulation has been used to measure distances for thousands of years.
6. The shape of the Greek letter delta is a triangle.
7. It is known that the sum of the three angles of a triangle always equals 180 degrees.
8. The largest angle of a triangle is always opposite the largest side and the smallest angle is always opposite the smallest side.

B. The triangle has three sides (it could be called a trigon) and three angles, yet there are many different parts associated with the triangle.

1. A line segment that connects a vertex to the side opposite the vertex and is perpendicular to that side is called an altitude.
2. A line segment that bisects one of the angles of a triangle is called an angle bisector.
3. A line segment that bisects a side of a triangle and is perpendicular to that side is called a perpendicular bisector.

4. A line segment whose end points are the vertex and the midpoint of a side is called a median.
5. A line segment whose end points are the midpoints of two of the triangle's sides is called a mid-segment.

C. Triangles are classified by the measure of the largest angle and by the relative sizes of the three sides.
1. If the largest angle in a triangle is greater than 90 degrees, it is an obtuse triangle.
2. If the largest angle in a triangle is equal to 90 degrees, it is a right triangle.
3. The other two angles in an obtuse or right triangle MUST be acute.
4. If the largest angle in a triangle is less than 90 degrees, it is an acute triangle.
5. If all the sides of a triangle are congruent, it is an equilateral triangle.
6. If only two of the triangle's sides are congruent, it is an isosceles triangle.
7. If the three sides are all different lengths, it is a scalene triangle.

D. Two triangles may have the same shape but a different size. These triangles are said to be similar.
1. Two triangles are similar if they have the exact same shape. The angles establish the shape of a triangle, so all corresponding angles must be congruent for the triangles to be similar.
2. If two angles of one triangle are congruent to the two corresponding angles in a second triangle, the third angle must be congruent, because the sum of the angles of any triangle must be 180 degrees. Thus, angle-angle is a valid proof of triangular similarity.
3. If two triangles are similar, then the corresponding sides must be in proportion. If two sides of a triangle are in proportion to the corresponding sides in a second triangle and the angle contained by the two sides in the first triangle is congruent to the angle contained in the two sides of the second triangle, then the triangles are similar. This is the side-angle-side proof.
4. Obviously, if all three sides of a triangle are in proportion to their corresponding sides in a second triangle, the two triangles are similar. This is a side-side-side proof.

E. Two triangles that have the same size and shape are said to be congruent.
1. Two triangles that have congruent corresponding sides are congruent. This is a side-side-side proof.
2. If the two triangles have two corresponding sides and the angles between those sides are congruent, then the triangles are congruent. This is a side-angle-side proof.

3. If two corresponding angles and the side between the angles are congruent, then the two triangles are congruent. This is an angle-side-angle proof.
4. Angle-angle-side is an alternate form of the angle-side-angle proof: if two angles are congruent, the third must also be congruent because all triangles have three angles, the sum of which is 180 degrees.
5. If the leg and hypotenuse of one right triangle are congruent to the leg and hypotenuse of a second right triangle, the two triangles are congruent. This hypotenuse-leg proof is based on the Pythagorean theorem.
6. Note that side-side-angle is not a method of proving all triangles congruent.

II. Trigonometry literally means "measuring with triangles." Knowledge of triangles had been used since ancient times to enhance the human ability to measure.

A. Knowledge about triangles was used to measure time thousands of years ago.
1. In the thirteenth century B.C., Egyptian mathematicians wrote about the use of shadows and proportional triangles to tell time.
2. Using triangles to measure distances was recorded in Mesopotamia in 1000 B.C., in Greece and China in 500 B.C., and in Sanskrit writings of the year A.D. 100.
3. Triangles were used to measure distances for navigation, but no formal study of the use of triangular relationships existed until the sixth century.

B. In the year 500 A.D., Indian mathematicians studying ideas about triangles written by ancient Greeks and Babylonians began to develop trigonometry.
1. The ratio of the side opposite a central angle in a circle and the hypotenuse of that triangle, which happened to be the circle's radius, was named the half chord. Because the radius of the circle was one and the opposite side was a half chord, the ratio between the opposite and the hypotenuse was the length of the half chord.
2. Because all right triangles with an acute angle equal to a specific central angle would be similar to the triangle in the circle, the ratio of the opposite to the hypotenuse for any similar triangle was known.
3. When the Arabs invaded India, they discovered the work being done in trigonometry and adapted it for their use. The term "half chord" was translated into Arabic. When this knowledge reached Europe, the Arabic word was mistranslated as "cove."
4. The Latin word for cove is *sinus*. Thus, we call the ratio between the opposite side and the hypotenuse of a right triangle the sine.

5. Because the ratio of the adjacent side to the hypotenuse is the sine of the complement of the angle, the ratio of adjacent to hypotenuse is called the cosine (complement sine).
6. If a right triangle is created whose two legs are the radius (with length one) and a tangent line, and a hypotenuse that is a secant line, we can find the names of the four remaining trigonometric ratios.
7. The ratio of opposite to adjacent is the tangent and the ratio of adjacent to opposite is the cotangent.
8. The ratio of the hypotenuse to the adjacent is the secant and the ratio of the hypotenuse to the opposite is the cosecant.

C. With tables of trigonometric values, measuring with triangles became a reality.
1. The heights of trees and mountains could be measured using tangent.
2. The distance across a river or a canyon could be measured using tangent.
3. The distance from a cannon to the walls of a city under attack could be measured using sine or tangent.
4. The angle required for a ramp could be measured using any of the trigonometric ratios, depending on what you knew about the ramp.
5. The length of a road for a certain grade could be calculated using sine or cosine.

III. With the use of analytic geometry, trigonometric ratios could be graphed and seen as functions.

A. Sine and cosine functions could be used to model situations that cycled or oscillated.
1. The oscillation of a pendulum could be modeled using a sinusoidal function.
2. Business and agricultural cycles can be modeled using a sinusoidal function.
3. Sound, light, and radio waves can be seen on an oscilloscope as sinusoidal functions.
4. Seasonal changes, such as temperature and rainfall, can be modeled with sinusoidal functions.

B. Tangent, cotangent, secant, and cosecant functions are not continuous and are not, therefore, as useful as sine and cosine functions in providing real-world models.

Essential Reading:

William Dunham, *The Mathematical Universe*, Utility.

NCTM, *Historical Topics for the Mathematics Classroom*, Sections IV and VI.

Supplementary Reading:

E. T. Bell, *Men of Mathematics.*

Howard Eves, *Great Moments in Mathematics (Before 1650).*

Jan Gullberg, *Mathematics From the Birth of Numbers.*

Morris Kline, *Mathematical Thought From Ancient to Modern Times*, Volume 1.

James R. Newman, *The World of Mathematics*, Volume One.

Sanderson Smith, *Agnesi to Zeno.*

Questions to Consider:

1. The trigonometric tables first developed in India were used to replace the table of chords developed in the second century by Claudius Ptolemy. What was the table of chords and what was its primary use?
2. Read the nineteenth-century satire "Flatlands" by Abbott. Explain how an isosceles triangle mutated to become an equilateral triangle. What was special about becoming equilateral?

Lecture Sixteen

Conic Sections

Scope: Conic sections is the name usually given to the study of parabolas, ellipses, and hyperbolas. The lecture will be divided into five sections. First we will examine the early history of the study of these shapes and the derivations of their names. The next three sections will be devoted to the three types of figures. The construction requirements and the general equation for each shape will be discussed. The lecture also offers a limited discussion of the applications for each shape. The final section will relate the three shapes to the path of a comet.

Outline

I. The descriptive terms "parabola," "ellipse," and "hyperbola" are Greek in origin.

A. The first person to use those terms to describe the conic section shapes was Apollonius.

1. His book *On Conic Sections* was published in 225 B.C.
2. He chose the terms from general mathematical terms used by the Pythagoreans.
3. Parabola was derived from *parabole*, the Pythagorean term for "placed beside."
4. Ellipse was derived from *elleipsis*, which meant "deficient or defective."
5. Hyperbola was derived from *hyperbole*, which meant "in excess."
6. The term conic section was used, because by slicing a cone with a plane, the cross section of the cone revealed by the slice would be an ellipse, a parabola, or a cone, depending on the angle of the slice.

B. Some of the work of Apollonius on conic sections was available to mathematicians of the sixteenth and seventeenth centuries.

1. The first four chapters of *Conic Sections* were translated into Latin in 1556. This was too late to be used by Nicholas Copernicus, who died in 1543.
2. Chapters five through seven were translated into Latin in 1661, which was after the deaths of Kepler and Galileo.
3. The last chapter of *Conic Sections* was lost.
4. The importance of conic sections became apparent once the idea that bodies traveled in circles was disproved by the work of Kepler and Galileo.
5. An object launched from earth will travel in a parabola and return to earth if its velocity is insufficient.

6. An object launched from earth with enough velocity to prevent return will travel in an elliptical orbit as a satellite of the earth.
7. An object launched with enough velocity to break the earth's hold will travel in a hyperbola through space.

II. Our first investigation will be of the ellipse.

A. An ellipse is a shape consisting of points, the sum of whose distances from the two foci is the same. As you travel in an elliptical path, the distance to one focus added to the distance to the other focus provides a constant sum.

1. The distance from the center of the ellipse to each focus is labeled "c."
2. The distance from the center to each vertex (turning point) on the major (longer) axis is labeled "a."
3. The distance from the center to each vertex (turning point) on the minor (shorter) axis is labeled "b."
4. For any ellipse $a^2 = b^2 + c^2$.

B. We will now examine the general equations for the ellipse whose center is at the origin.

1. If the major axis is the x-axis, the equation is $x^2/a^2 + y^2/b^2 = 1$.
2. If the major axis is the y-axis, the equation is $x^2/b^2 + y^2/a^2 = 1$.
3. Remember that a is always greater than b. A glance at the equation will reveal whether the ellipse is oriented horizontally or vertically.
4. If a and b are equal, then the ellipse is a circle and the two foci collapse into the center.

C. The major use of the ellipse is to trace the path of a satellite.

1. Using Tycho Brahe's data on planetary positions, Kepler was able to state that the paths of the planets were ellipses and he was able to formulate three laws that described planetary motion with precision.
2. The elliptical orbits were to be expected for any body traveling about a larger body, such as a moon about its planet.
3. In the latter half of the twentieth century, we began to place artificial satellites in orbits around the earth. These orbits are all ellipses.

III. Next we will examine the parabola.

A. A parabola consists of points whose distance from a focus point is equal to the perpendicular distance to a line called the *directrix*.

1. The equation of a vertical parabola is $y = a(x - h)^2 + k$, where the vertex is located at (h, k).
2. The equation of a horizontal parabola is $x = a(y - k)^2 + h$, where the vertex is located at (h, k).
3. The vertex is located halfway between the directrix and the focus.

B. The parabolic shape has many uses, but the ballistic trajectory is probably the most significant.
   1. All bodies traveling in free fall travel in a parabola. This is true for a ballistic missile (once its engine is turned off), a kicked or thrown ball, a bullet, and many other examples.
   2. A three-dimensional parabola, a paraboloid, reflects incoming rays to its focus. Thus, the shape of many reflecting dishes (for radar or satellite TV) is the paraboloid.
   3. The parabola is the graph of a quadratic function. We have seen in an earlier lecture that the quadratic model is both useful and common.
   4. In 1100, Omar Khayyam used parabolas to obtain solutions to cubic equations.

IV. Our last conic section to be discussed is the hyperbola. We will restrict ourselves to hyperbolas whose centers are at the origin.

A. A hyperbola is the set of all points where the difference between the distances from the two foci is constant. A hyperbola is comprised of two parts that are symmetric.
   1. Between the two parts of the hyperbola is a rectangular space. The two vertices of the hyperbola are on the rectangle.
   2. The diagonals of the rectangle are the asymptotes that form the boundaries for the hyperbola.
   3. The distance from the center to each focus is labeled "c."
   4. The distance from the center to the vertex is labeled "a." This is also the distance along the major axis to the rectangle.
   5. The distance along the minor axis from the center to the rectangle is labeled "b."
   6. For any hyperbola, $c^2 = a^2 + b^2$.

B. There are three versions of the general equation of the hyperbola.
   1. If the hyperbola has the x-axis as its major axis, it is oriented horizontally, and the equation is $x^2/a^2 - y^2/b^2 = 1$.
   2. If the hyperbola has the y-axis as its major axis, it is oriented vertically, and the equation is $y^2/a^2 - x^2/b^2 = 1$.
   3. An equation of the form $xy = k$, where k is a constant, will have a graph that is a hyperbola.

C. The hyperbola does not have as many applications as the ellipse and the parabola.
   1. Loran, an ocean navigation system, uses hyperbolas to plot locations.
   2. The shapes of the cooling towers at nuclear power facilities are hyperboloids (three-dimensional hyperbolas).
   3. The $xy = k$ hyperbola is a graphical representation of indirect variation, such as the relationship between wave frequency and wavelength or gas pressure and volume under constant temperature.

V. The path of a comet, as studied by Edmond Halley in 1704, depends on its velocity. In predicting the shape of the path of the comet, we are concerned about three values other than the comet's velocity.

A. Those values are symbolized by G, M, and p.
   1. G is the universal gravitational constant.
   2. M is the mass of the sun.
   3. The symbol p stands for the distance from the center of the sun to the vertex of the comet's path.

B. When the velocity of the comet at its vertex is compared to the square root of 2 times the product of G and M divided by p, the result dictates which conic section represents its path.
   1. If the velocity is less than the square root, the path is an ellipse.
   2. If the velocity is exactly equal to the square root, the path is a parabola.
   3. If the velocity is greater than the square root, the path is a hyperbola.
   4. Only a comet whose path is an ellipse (such as Halley's comet) will be seen a second time on earth.

Essential Reading:

Roland E. Larson, *Precalculus with Limits—A Graphing Approach*, Chapter 10.

NCTM, *Historical Topics for the Mathematics Classroom*, Section IV.

Sanderson Smith, *Agnesi to Zeno*.

Supplementary Reading:

E. T. Bell, *Men of Mathematics*.

Morris Kline, *Mathematical Thought From Ancient to Modern Times*, Volume 1.

James R. Newman, *The World of Mathematics*, Volume One.

Questions to Consider:

1. In his investigation of comets, Halley had evidence that there had been a comet seen in England in 1066. What work of art provided this evidence?
2. What characteristic of the electromagnetic radiation that is reflected by a parabolic telescope causes the radiation to be focused at the focus of the paraboloid?
3. Sketch the graph of the hyperbola $xy = 12$ (do not forget about the negative values, such as (-3, -4)). How does that graph compare to a vertical or horizontal hyperbola?

Lecture Seventeen

Introduction to Probability

Scope: This is the first of three lectures on probability. In this lecture, we will first examine the history of the development of probability. Next we will make a thorough investigation of the rules of probability—the three fundamental rules of OR, AND, and NOT will be emphasized. A simple example will be used to illustrate the application of the three basic rules of probability. The last part of this lecture will be devoted to the applications of probability in our lives. Examples are drawn from business, meteorology, sports, and the military.

Outline

I. Probability is a measure of uncertainty. There is much evidence that probability concepts were understood in ancient times.

A. Those who foretold the future were aware of probabilistic relationships, and gambling became the vehicle for a mathematical investigation of probability.

1. Soothsayers used the knuckle bones of dogs much as dice are used today. Written records of the meanings of the various throws indicate that those who used the bones understood the probability of these outcomes.
2. Recognizing that gambling involved chance that could be measured led to the beginnings of the mathematical investigation of probability.
3. In 1494, Luca Pacioli wrote about games of chance.
4. During the sixteenth century, Gerolamo Cardano studied gambling and used his mathematical training to quantify the rules of gambling.
5. Cardano published his findings after he used his knowledge to accumulate wealth by winning.
6. During the seventeenth century, Blaise Pascal and Pierre de Fermat conducted an exchange of correspondence about the outcomes of the rolls of dice.
7. In 1718, Abraham De Moivre wrote *Doctrine of Chance*, which investigated the theory of probability.
8. Later in that century, G. L. Leclerc, the Comte de Buffon, flipped a coin 4040 times and recorded that 2048 of the outcomes were heads. Based on this experiment, the probability of getting heads was .5069, which was close to the theoretically assumed probability of one-half.
9. Buffon's experiment was the first of many attempts to validate the law of large numbers, which states that if you repeat an event enough times, the experimental result should match the theoretical result.

B. Little was done in terms of the scientific investigation of probability until the twentieth century.

1. At the beginning of the century, Karl Pearson, one of the fathers of statistics, repeated Buffon's experiment by flipping a coin 24,000 times. He counted 12,012 heads for a probability of .5005.
2. John Kerrich, an Australian mathematician, imprisoned by Germany during World War II, used his incarceration to flip a coin 10,000 times. He obtained 5067 heads, which translates into a probability of .5067.
3. The expanding need for insurance during the twentieth century required a scientific study of the results of chance as part of actuarial science.
4. As businesses grew and the operation of major businesses became more complex, there was an increasing need to quantify the chance of success for various strategies. From this need, a branch of mathematics called game theory was born. Game theory is a probabilistic treatment of win-lose situations.

II. Because probability is a branch of mathematics, some basic definitions and fundamental rules must be established.

A. Probability is a measure of uncertainty. There are two types of certainty—you can be certain that an event will happen or you can be certain that an event will not happen.

1. An event that is certain to happen, such as rolling a standard six-sided die and obtaining a number less than seven, is said to have a probability of one.
2. An event that is certain not to happen, such as flipping a standard U.S. quarter with the result that the side showing has a bust of King George III on it, is said to have a probability of zero.
3. Because all degrees of uncertainty lie between the two forms of certainty, the probability for an uncertain situation will be between zero and one, i.e., a proper fraction.
4. If all possible outcomes are equally likely, the probability of a particular result will be the ratio of the number of outcomes that will provide the indicated result to the total number of outcomes.

5. As an example we list all the possible outcomes for the flipping of three coins (a penny, nickel, and dime).

Penny	Nickel	Dime
H	H	H
H	H	T
H	T	H
H	T	T
T	H	H
T	H	T
T	T	H
T	T	T

6. Because each outcome is equally likely if we assume that the coins are fair, there are eight possible outcomes.
7. The result of two heads occurs for three of the outcomes. Thus, the probability of getting two heads must be three-eighths or 37.5%.
8. Because it is certain that one of the results will occur, the sum of the probabilities of the various results must be 1.
9. In the example of the three coins, the sum of the probabilities of getting zero heads, one head, two heads, and three heads is 1/8 + 3/8 + 3/8 + 1/8 = 1.

B. The three basic rules of operating with probabilities involve AND, OR, and NOT.
1. The probability of selecting a face card or a seven from a standard deck of 52 playing cards is the sum of the two probabilities (P(A OR B) = P(A) + P(B).
2. If we seek the probability of selecting a face card or a heart, that formula does not work because we are double-counting the jack, queen, and king of hearts.
3. The probability of A OR B is equal to the sum of P(A) plus P(B) minus the probability of both happening (P(A AND B)).
4. If A and B cannot occur simultaneously (such as a card being both a face card and a seven), then the outcomes are said to be mutually exclusive or disjoint. The P(A AND B) in that situation is zero.
5. If two events have no effect on each other and the probability of one is not changed by whether the second event occurs, these events are said to be independent. Symbolically, this is written P(A) = P(A|B). P(A|B) is the conditional probability of A occurring if B occurs.
6. If two events are independent, the probability of both occurring, P(A AND B), equals the product of P(A) and P(B).
7. If two events are not independent, then the AND probability is the product of the probability of one and the conditional probability of the second occurring if the first occurs: P(A AND B) = P(A) x P(B|A).
8. Remember: AND goes with addition; OR goes with subtraction.
9. Because the sum of the probabilities of all outcomes must be 1, the sum of the probability of an event occurring and the probability of the event not occurring must be 1: P(A) + P(NOT A) = 1.
10. The event "NOT A" is called the complement of A.

C. An example is used to demonstrate the computation of a probability involving OR, AND, and NOT.
1. A customer enters a book and music store. She has a 1/4 probability of purchasing a book and a 2/5 probability of purchasing a CD. The two events are independent.
2. The question that is asked is what is the probability that she buys neither a book nor a CD?
3. Because the event "buying nothing" is the complement of "buying something," the probability we seek is 1 - (P(Book OR CD)).
4. The P(Book OR CD) = P(Book) + P(CD) - P(Book AND CD) from our formula for the OR probability.
5. Because the purchase of a book and the purchase of a CD are independent events, P(Book AND CD) = P(Book) x P(CD).
6. Putting all this together gives us P(buying nothing) = 1 - [P(Book) + P(CD) - [P(Book) x P(CD)]].
7. The answer is 1 - (1/4 + 2/5 - 1/4 x 2/5) = 1 - 11/20 = 9/20. There is a 45% chance that she will buy nothing.
8. We could also obtain the answer by computing the product of the two complements. Multiplying 3/4 x 3/5 also gives an answer of 9/20. This is an example of De Morgan's law.

III. Probability is used in sports, insurance, business planning, the military, and meteorology.

A. Sports, and gambling on sports, provides a long list of the uses of probability.
1. Some examples in baseball are batting average, fielding average, and winning percentage.
2. In basketball, field goal accuracy, free throw accuracy, and three-point shot accuracy are all measured with percentages that are probabilities.
3. In football, passing completion percentage, third-down efficiency, and balance between running and passing plays are measured with percentages that are probabilities.

B. The decision-making process in life and casualty insurance, which is called underwriting, is based on probabilities.
1. Your age, health, occupation, and whether you smoke or not are factors in determining the cost of life insurance.

2. The nature of the structure in which you live, the use of burglar and smoke alarms, the location of the nearest fire station and the nearest fire hydrant are used to forecast the probability of the insurance company taking a loss on your homeowner's or renter's insurance.
3. Age, gender, type of vehicle, and driving record are all factors in determining the cost of automobile insurance. For a high school or college-age male, the cost of automobile insurance can be lowered by maintaining a B or better average in school, completing a driver education course, or getting married. All affect the probability of the driver causing an accident that will result in a loss for the insurance company.
4. Managers of business use probability in determining locations of retail outlets, in making decisions on new products, and in designing marketing strategies to improve sales.
5. The military uses probability to predict weapon effectiveness against certain types of targets, to evaluate particular strategies, and to predict failure of components in weapon systems.
6. In meteorology, the chance of rain is a probability, and forecasting the appearance of tornadoes and the movement of hurricanes uses probabilistic tools.

Essential Reading:

David S. Moore and George P. McCabe, *Introduction to the Practice of Statistics,* Chapter 4.

Supplementary Reading:

NCTM, *Historical Topics for the Mathematics Classroom*, Section IV.

Sanderson Smith, *Agnesi to Zeno.*

Questions to Consider:

1. Find the probability of O-ring failure on the shuttle flight before the Challenger disaster when the outside air temperature was less than 50° F. Given that at the time of the Challenger launch, the temperature was below freezing, why was the launch made?
2. Research the probabilities of the various hands in poker. Can you use those probabilities to determine the hierarchy of winning hands?
3. If I flip a coin twenty times and obtain fifteen heads, is that an indication that the coin is not fair? Why or why not?

Lecture Eighteen

Binomial and Normal Probability

Scope: The two most commonly used types of probability distributions are the binomial and the normal. This lecture will provide a thorough examination of the characteristics and applications of these probability distributions. The binomial distribution is used in situations in which we are limited to two outcomes, such as success/failure, win/lose, profit/loss, disease cured/disease not cured, and pass/fail. The binomial probability formula is developed and Pascal's triangle is constructed and used for binomial computations. The normal distribution is given that name because it is found in all aspects of life. Physical measurements, such as height and weight, are normally distributed. So are psychological measurements, such as IQ. The weights of packages coming off a production line are normally distributed, as are the measurements of the saltiness of potato chips. Of equal importance, the normal distribution is a key distribution in our use of inferential statistics. The use of the standard normal table is explained and demonstrated. This table will be used in future lectures on hypothesis testing and surveys.

Outline

I. There are three strict requirements for a distribution to be binomial, and there are many situations that meet those requirements. Once we understand the characteristics of a binomial distribution, we can obtain the binomial probability formula.

   A. There are three requirements for a distribution to be binomial.

      1. There must be only two possible outcomes. This is called a Bernoulli population, named after Jacob Bernoulli, who studied this distribution in the latter part of the seventeenth century.
      2. The two outcomes are identified as "success" and "failure." In this situation, these terms carry no emotional attachment.
      3. The second requirement is that each event in a binomial trial is independent of the other events in the trial. The probability of success (symbolized as p) and the probability of failure ($q = 1 - p$) do not change from event to event.
      4. The third requirement is that the number of events (n) in one trial is fixed. Typically, at the end of the trial, the number of successes is counted.

   B. There are many situations that have binomial probability distributions.

      1. Genetics, with its concern about dominant and recessive factors, offers many binomial distributions.

2. The world of sports offers many binomial opportunities, such as win/lose, ball/strike, score/do not score, make the shot/miss, and penalty/no penalty.
3. In medicine, the outcomes of surgery and medical treatment are binomial situations.
4. Politics, being a win-lose proposition, has many uses for the binomial distribution.
5. Businesses faced with decisions about products, personnel, and strategies must use binomial outcomes to aid in making those decisions.
6. In our personal lives, we are faced with success/failure decisions about marriage, vacation, employment, relocation, and many other situations that are binomial in nature.

C. The binomial probability formula can be determined after observing a simple binomial situation.

1. Let's say a student takes a three-question multiple-choice test, and each question has a choice of four answers. If only one answer is correct for each question and the student must guess for each question, this is a binomial situation.
2. Because the test has three questions, n = 3. Because the chance of guessing the right answer is one out of four, p = .25 and q = .75.
3. To determine the probability of a specific number of correct answers, the number of outcomes that will have a particular result, the probability of getting the specific number of correct answers, and the probability of getting the remaining answers wrong are all needed. We symbolize the number of successes with the letter x.
4.

x	Number of Outcomes with x	P (x successes)	P(n – x) failures
0	1	0	.75 x .75 x .75
1	3	.25	.75 x .75
2	3	.25 x .25	.75
3	1	.25 x .25 x .25	0

5. The number of outcomes leading to a particular result is symbolized using nCx, the number of "combinations" providing x successes in n events.
6. Based on our problem, the binomial probability formula is P(x) = nCx $p^x q^{(n-x)}$.

D. We need a way to find nCx, and Blaise Pascal provided a means with the triangle named after him.

1. Pascal's work in which he described the triangle was published in 1665, three years after his death.

2. Although the triangle is named for him, Pascal did not discover it. It was known to Omar Khayyam in 1100 and to the Chinese in 1303.
3. The first few rows of the triangle look like this:

```
                  1
                1   1
              1   2   1
            1   3   3   1
          1   4   6   4   1
        1   5  10  10   5   1
      1   6  15  20  15   6   1
```

4. If we write Pascal's triangle as a right triangle, its appearance is more useful for binomial probability computations:

n/x	0	1	2	3	4	5	6
0	1						
1	1	1					
2	1	2	1				
3	1	3	3	1			
4	1	4	6	4	1		
5	1	5	10	10	5	1	
6	1	6	15	20	15	6	1

5. The number of ways to get four successes from six events is fifteen according to our table.

II. The normal probability is of great interest because of utility. The normal distribution is represented by a specific exponential function. Work on this function has been going on for nearly 300 years.

A. The normal probability distribution is defined by the following exponential function:

$$P(x) = (1/\sigma \sqrt{2\pi})\, e^{-.5[(x - \mu)/\sigma]^2}$$

1. P(x) is the probability of getting the outcome x.
2. The population mean is labeled μ and the population standard deviation is labeled σ.
3. The symbol π is the same as it was in our discussion of circles.
4. The earliest analysis of this exponential function was done by Abraham de Moivre in his book *Doctrine of Chance*, which was published in 1718.

5. Carl Gauss extended this work at the beginning of the nineteenth century. Because of his work, the normal distribution is often referred to as the Gaussian distribution.
6. The graph of this function is symmetric and is shaped like a mound or a bell. When somebody mentions the bell curve, he or she is referring to the normal distribution.

B. The normal distribution is a continuous function and computations of probabilities are made using integral calculus.
   1. Because the distribution is continuous, the only types of probability computations that can be made are the probability of the result being greater than x, less than x, or between two values of x. Theoretically, the probability of a specific number is zero.
   2. Many uses of the normal distribution are for distributions that are not continuous but discrete. Adjustments are easily made to allow the normal distribution to be used as an accurate approximation.
   3. All normal distributions are symmetric about the mean, no matter what the mean and standard deviation are. The mean, the median, and the mode of the distribution are the same in a normal distribution.
   4. For any normal distribution, the probability of being in the interval from one standard deviation below the mean and one standard deviation above the mean is 68%.
   5. The probability of being within two standard deviations of the mean is 95% and within three standard deviations, the probability is 99.7%.
   6. The 68 - 95 - 99.7 rule is true for any normal distribution. These sections defined by standard deviation are quite significant when categorizing people based on IQ.

III. The development of the Central Limit theorem made the normal distribution so widely used that one easy-to-use table had to be devised to be used in any normal situation.

A. The Central Limit theorem defined the characteristics of the sampling distribution of the means of the large number of samples of the same size that can be drawn from a given population.
   1. The first work on the Central Limit theorem (CLT) was accomplished by Simon Laplace in the early part of the nineteenth century.
   2. The CLT for the sampling distribution of sample means says that if the population standard deviation is known and if the sample size is large enough, the sampling distribution will be approximately normal. The distribution will become more normal as the sample size increases.

   3. The CLT is true no matter what the underlying distribution of the population is. If the population itself is normal, then the sampling distribution is normal.

B. The Standard Normal table or z table was developed so that only one table is required when working with a normal distribution. (Note: A z-table is provided for your reference in the appendix, page 74)
   1. The value of z is the number of standard deviations the measurement is from the mean [$z = (x - \mu)/\sigma$].
   2. The z table provides the probability of a measurement being less than x. This is simply the area under the curve to the left of z, which was found using the definite integral.
   3. If the probability of a measurement being greater than x is desired, one must subtract the table's probability value from 1.
   4. If x is below the mean, the z score will be negative. If the z score is 0, then the value of x is the same as the mean and, because of symmetry, half of the probability is below the mean.
   5. To find the value of x that will cut off a particular amount of probability, one must first determine the probability to the left of the unknown measure. For example, if you want the cutoff for the top 10%, the probability to the left will be .9 since 1 - .1 = .9.
   6. Entering the z table with the probability, the z score is obtained. Rearranging $z = (x - \mu)/\sigma$ we solve for x, $x = \mu + z\sigma$.

C. A simple example is provided to demonstrate the use of the z table.
   1. A particular graduate admission test has a mean score of 600 with a standard deviation of 120.
   2. If a person scores 780 and desires to know the percentile for that score, the z table is used.
   3. The z score is (780 - 600)/120, which is 1.50. According to the z table, $P(X < 780)$, which in this case is the same as $P(z < 1.50)$, which is .9332. The person is in the 93rd percentile, and about 7% of those taking the test will get a higher score.
   4. Your boss takes the test and receives a score of 510. The z score for that grade is -0.75. The boss is in the 22nd percentile.
   5. A particular graduate school will not consider an applicant unless the applicant's score on the admissions test is in the top 10%. The question is, what is the minimum score required for consideration?
   6. As was shown earlier, we enter the table with a .9000 probability and obtain a z score of 1.28, which gives a test score of 600 + 1.28(120) = 753.6. A score of 754 or higher will meet the graduate school's criterion.
   7. We can use a normal approximation to solve probability questions for binomial distributions provided that the products of n times p and n times q are each at least 10.

Essential Reading:

David S. Moore and George P. McCabe, *Introduction to the Practice of Statistics*, Chapters 1 and 5.

Sanderson Smith, *Agnesi to Zeno*.

Supplementary Reading:

NCTM, *Historical Topics for the Mathematics Classroom*, Section IV.

Questions to Consider:

1. How is a permutation different than a combination? Find the formula for nPx and compare it to the formula for nCx. What is the relationship between the formulas?
2. Given the use of Pascal's triangle in the binomial probability computation, why must the triangle be symmetric?
3. Research the Student t table. How is it different from the standard normal table? When is each used? The mathematician who derived the table was William Gosset. Why did he use Student rather than his real name?

Lecture Nineteen

Probability and Simulation

Scope: With the increasing effectiveness and speed of technology, the use of simulations to provide information in problem-solving situations has accelerated. The simulations discussed in this lecture use probability information to gather data to answer a question or solve a problem. The simulations demonstrated here will be simplistic to foster learning. Once you understand the purpose and structure of a simulation, you can use the power of technology to solve more complex problems. The lecture starts with a rationale for simulation and some examples of applications. The key step in any data-gathering activity is design, and a basic procedure for designing a simulation is discussed. Simulations for the three types of Bernoulli trials are investigated, followed by a brief look at the design for a few more complex situations. The last activity is a look at how a simulation could be used to test a hypothesis in a common real-world situation.

Outline

I. In our complex society, there is a need for a method of creating data based on some knowledge about an environment so that decisions could be made based on the analysis of the data.

- **A.** Simulations are effective because they are relatively inexpensive, present little chance of actual danger, and can be repeated as many times as desired.
    1. A non-mathematical simulation that demonstrates the advantages of cost, safety, and repetition is the airline flight simulator.
    2. Pilots must receive practice in emergency procedures, yet it would be impractical to have a group of pilots take an airplane into a thunderstorm and turn off all the engines or shut down the electrical supply so that they could gain experience in overcoming the problems presented.
    3. A aircraft simulator, while rather expensive, is much less expensive than the cost of replacing airplanes that were lost during practice flights in thunderstorms.
    4. Flying an airplane into a thunderstorm to train pilots in emergency procedures would place all aboard that airplane in great danger. A flight simulator is a much safer environment.
    5. If the pilots were successful in their first attempts at overcoming emergency conditions, they would probably not relish having to repeat the situation over and over. In a flight simulator, pilots can experience repeated opportunities to deal with realistic emergency conditions.

B. Mathematical simulations using information about probability can be equally effective in providing useful data for decision making.

1. Traffic flow on city streets or on highways can be simulated with numerical values.
2. Business strategies can be tested using simulations based on probability information available from past performances or through sampling.
3. Tactics in various sports can be simulated using team tendencies and success rates.
4. Weather can be simulated mathematically using probability information gathered in seasons past.
5. Medical experiments can be simulated using data from past experiments.

II. The key to the success of a simulation is the design process. This is a simple five-step process that provides a foundation for understanding how the design process works in more complex simulations.

A. The process requires that complete information on types of outcomes, probability of outcomes, and the nature of the trial be known before starting the five-step process.

1. The first step is to state the nature of a simple event in the process and define the possible outcomes for a simple event. An example of a simple event might be one sales call with the possible outcomes being no sale, a sale of $100 worth of product, or the sale of $1000 worth of product.
2. The second step in the design process is to quantify the probabilities for each possible outcome defined in the first step. In our example, $P(0) = .6$, $P(100) = .37$, and $P(1000) = .03$. The sum is 1.0.
3. The next step is to select a probability tool that would simulate the probabilities indicated in the previous step. In this case, we might use random numbers from 1 through 100. Numbers 1 through 60 would represent no sale, 61 through 97 would represent a $100 sale, and 98 through 100 would represent a $1000 sale.
4. It is vital that each of the outcomes to the simulation tool in step three have the identical probability as the outcome that it is simulating.
5. The next step is to define how many simple events complete one trial. In our example, we expect our sales people to make five sales call in a typical day. We would say that $n = 5$.
6. The final step is to define what is to be counted or measured and recorded at the conclusion of each trial. In the example, we could count the number of each type of sale or we could record the total sales for each day simulated.

B. Because the value of a simulation depends on the ability of the selected probability tool to replicate the outcomes being simulated, choosing the tool can have significant consequences.

1. If the simulation involves two outcomes, each with a 50% probability of occurring, a fair coin could be used.
2. If the probabilities are all fractions for which six could be a common denominator (denominators of 2, 3, and/or 6), a standard fair die could be used.
3. Spinners can be designed to produce the required probabilities.
4. A printed random 10-digit table could be used to simulate any decimal probability. Digit tables can be used to simulate almost any probability if the appropriate digits are used and other digits are disregarded.
5. A random number generator on a calculator or computer can be used to generate strings of random numbers to meet the needs of the simulation.

III. A basic situation for simulation is a Bernoulli trial.

A. A Bernoulli trial requires two outcomes in which each repetition of an outcome is independent of all the other outcomes for that trial.

1. The two outcomes are typically labeled as success and failure.
2. Because the events are independent, the probability of success (p) and probability of failure (q) do not change during the trial.
3. The Bernoulli trial was named after Jacob Bernoulli, who analyzed this type of probability situation around the year 1700 in a book entitled *Ars Conjectandi*.
4. The three types of Bernoulli trials, binomial, geometric, and negative binomial, can all be easily simulated.

B. As was defined in a previous lecture, a binomial trial is a Bernoulli trial with a fixed number of simple events. A simple binomial situation to simulate could involve finding the probability of a family of five children containing exactly three girls.

1. A simple event in this problem is the birth of one child. The possible outcomes are a boy or a girl.
2. The probability of having a boy child is assumed to be .5, as is the probability of the child being a girl. The sex of each child is independent of the sex of the other children in the family. The probability of a girl is always .5.
3. A coin will be used to simulate a birth. Heads will simulate a girl and tails will represent a boy. Note that the probability of heads is the same as the probability of a girl being born.
4. A trial will be made up of five births.
5. At the conclusion of the trial, the number of girls in the family will be counted and recorded.

C. A geometric trial is a Bernoulli trial that is stopped when the first success is obtained. A simple geometric situation that can be simulated involves answering the following question: On average, how many children will a family have until the first girl is born?
1. Step one identifies the simple event as the birth of a child with the outcomes being girl and boy.
2. In step two, the probability of a boy being born is defined to be .5 and the probability of a girl is also .5.
3. Step three assigns the use of a coin as the probability tool with heads representing a girl and tails representing a boy.
4. Step four defines a trial to end when the first girl child is born (the first head is flipped).
5. Step five would state that at the conclusion of each trial, the number of children in the family (the number of simple events) will be counted and recorded.

D. A negative binomial trial is a Bernoulli trial that is concluded when a certain number of successes is obtained. A simple negative binomial situation that can be simulated involves finding how many children would have to be born, on average, until there were three girls in the family.
1. Noted that the geometric trial is a very special case of the negative binomial trial in which the number of successes required is one.
2. The steps for the negative binomial situation are the same as for the geometric simulation with the exception of step four.
3. Step four would define the trial as concluding when the number of girls in the family is three.

IV. Not all situations that can be simulated are Bernoulli and not all Bernoulli situations are simplistic.

A. There are many common situations that do not have only two outcomes and/or do not display independence.
1. The simulation of a baseball world series can be easier to do than the mathematical computations involved in calculating the probability of the series lasting seven games. The simulation might be Bernoulli, but it cannot be classified as one of the three types, because there is no fixed number of games played but the series must terminate by the end of the seventh game.
2. Today's weather is the consequence of yesterday's weather, so the probability of success will change, negating the Bernoulli requirement for independence.
3. In a predator-prey simulation, the probabilities for survival of both types of life forms are dependent on the population size of each. Thus, there is not independence. This is a reasonable and important situation to simulate, but it is not a Bernoulli trial.

4. The sales simulation used earlier in the lecture is not Bernoulli because there were three possible outcomes for each sales call (simple event).

B. An example of a powerful yet simple use of a Bernoulli simulation involves a question of prejudicial hiring.
1. A company is planning to hire 30 technicians. These are highly skilled and well-paid employees.
2. It is determined that 40% of the qualified applicants are female; therefore, it is expected that the company will hire 12 female technicians.
3. When the hiring process is concluded, only 8 female technicians were hired; the remaining 22 technicians were male.
4. Immediately, there are accusations of prejudicial hiring practices. An insufficient number of females hired seems to indicate a bias against hiring female technicians.
5. Rather than argue about what might have happened, we can design a simulation and run many repetitions to determine a pattern of results. Because the simulation will be based on random selection (no bias), the simulated results should give a clue as to whether there is evidence of prejudice in the employment procedure.
6. This is a binomial situation with $p = .4$ and $n = 30$. Each hire is independent of the others, so the design is quite simple.
7. I conducted 1000 simulations of this trial with the following results (only trials that produced 8 or fewer female technicians hired are shown).

Number of Female Techs. Hired out of 30	Number of Simulations with That Result
4	1
5	4
6	12
7	26
8	50

8. The results of 1000 simulations show that a total of 93 trials had 8 or fewer females hired when the results were due strictly to chance. That represents 9.3% of the simulations.
9. The question that must be answered is: Is 9.3% a rare occurrence or is it relatively common? If 9.3% is rare, then the actual outcome has a small probability to occur by chance, which means that the simulation results indicate that there was hiring bias.
10. If 9.3% is too large a percent to be considered rare, then the simulation fails to demonstrate hiring bias.
11. In most statistical tests, the largest percentage that would be considered "rare" is 5. The results from the simulation seem to

indicate that bias is unlikely. As we shall see in the lecture on hypothesis testing, this process rarely leads to a certain conclusion.

Essential Reading:

Mrudulla Gnanadesikan, Richard L. Scheaffer, and Jim Swift, *The Art and Techniques of Simulation.*

Supplementary Reading:

Frederick Mosteller, Robert E. Rourke, and George B. Thomas, Jr., *Probability: A First Course.*

NCTM, *Historical Topics for the Mathematics Classroom*, Section IV.

Questions to Consider:

1. Assume that the technician-hiring situation had occurred in your town. Write a "simulated" letter to the editor of your local newspaper using the simulation results to argue for or against evidence of prejudicial hiring practices.
2. If a person showed you the result of twenty-five simulations and asked you to trust the results, would you? Why or why not?
3. Show how the Law of Large Numbers would be used to recommend conducting the maximum number of simulations possible.

Lecture Twenty

Data Displays and Numerical Statistics

Scope: This is the first of four lectures covering statistics or, more precisely, data analysis. Statistics is really not a branch of mathematics. Rather, it is a social science that is an intense user of mathematical tools. This introductory lesson has three goals. First, we will attempt to define statistics and take a brief look at the history of its development, most of which happened during the last 100 years. Then, we will look at simple numerical statistics and graphical displays that are used in the analysis of a single variable. Finally, we will gain some insight into the importance of data analysis by briefly considering its applications.

Outline

I. Data have been around for a long time but little was done in terms of analysis of data until the middle of the nineteenth century. Statistics is a social science that uses mathematical tools to analyze data.

- **A.** An ever-increasing number of colleges are recognizing that statistics is not a branch of mathematics by having a statistics department that is totally independent of the mathematics department.
  1. Statistics involves the collection, display, and analysis of data.
  2. In this lecture, we are focusing on the exploratory data analysis (EDA) of a single variable.
  3. Data can be gathered from a single source or from various sources. In the latter case, the analysis will involve a comparison of the results from the different sources.
  4. EDA is qualitative in nature. We are concerned with measures of tendency, variation, pattern, and extreme values.
  5. Once EDA is understood, work can be done in confirmatory data analysis, which is also known as inferential statistics. Lectures Twenty-Two and Twenty-Three will investigate this area of data analysis.
- **B.** The effective analysis of data began in 1857, although the analysis was disregarded and cost the lives of many British soldiers. The importance of data analysis increased throughout the latter part of the nineteenth century and into the twentieth century.
  1. In 1857, Florence Nightingale conducted an analysis of the mortality rate of British soldiers fighting in the Crimean War.
  2. Her data analysis seemed to indicate that more casualties were dying from lack of proper sanitation than from the severity of their wounds.

3. Despite the amount of data she had gathered, the professional manner in which she had presented her analysis, and the critical nature of her findings, her work was disregarded by the British Army command because she was a woman—this was the middle of the Victorian era.
4. By the end of the nineteenth century, serious work was being done to bring scientific method to the analysis of data. Karl Pearson heads the list of statisticians who helped formalize the use of data in making decisions. In addition to his coin-flipping experiment mentioned in a previous lecture, Pearson invented the chi square statistics, the oldest inferential statistics test (it is still in use today). He also was a significant contributor to the study of correlation, a subject to be discussed in our next lecture.
5. During the first half of the twentieth century, Sir R. A. Fisher provided major contributions to the science of data analysis. He is given credit for inventing statistical experimental design and the formal methods to analyze the data obtained from such experiments.
6. During the latter half of the twentieth century, John Tukey was the guru of data analysis. He sought graphical displays with "inter-ocular impact" (i.e., he wanted the displays to hit you between the eyes!). Perhaps his most widely used creation is the box plot, a graphical display that we will examine shortly.

II. There are many numerical statistics and graphical displays available for the statistician. We will look at a few of the useful examples.

A. A graphical display should allow the viewer to get some sense of pattern about the data distribution. Also, the nature of extreme values should be evident and an estimate of a representative "average" number should be apparent from the image. A sense of the amount of variation should be obtained by looking at the display.
   1. A line plot is a simple and effective data display. A number line is constructed and X marks are made above the number line to indicate the frequency of occurrence of the various values.
   2. A line plot is an excellent tool for identifying clusters of data and gaps between data, as well as potential outliers (data that are far enough removed from the bulk of the data to raise questions).
   3. A histogram allows the counts for various classes of data to be viewed. The height of each bar is the frequency of data falling into that class. The histogram is an excellent device for assessing the pattern inherent in the data distribution. Is the distribution fairly symmetric or are the data skewed (stretched) in one direction?
   4. A box plot provides a visual representation of a measure of tendency and two measures of variation and can be used to analyze pattern and identify outliers.

B. The numerical statistics that we will examine are measures of tendency and variation.
   1. The two most common measures of tendency are the mean and the median. The mean is obtained by adding up all the data and dividing by the number of pieces of data. The median is the measurement in the middle of the data once they are placed in numerical order.
   2. The mode, or value that occurs the most, is another measure of tendency. It is most useful when analyzing non-numerical data, because the mean and median are useless in those situations.
   3. The quartiles are the measurements at the 25th percentile and at the 75th percentile. The lower quartile is the midpoint of all the data below the median and the upper quartile is the midpoint of all the data above the midpoint.
   4. The difference between the quartiles (UQ - LQ) is called the inter-quartile range or IQR. It is a measure of variation that shows the spread of the middle 50% of the data. It is unaffected by extreme values.
   5. The range is the difference between the minimum and maximum data values. It is a measure of variation and is totally affected by the extremes.
   6. The standard deviation is a measure of variation that is somewhat affected by extreme values. If we recall our discussion of the normal distribution, the standard deviation is a key measurement in working with that probability distribution.
   7. The box plot is constructed using what is called the five-number summary: the minimum value, the lower quartile, the median, the upper quartile, and the maximum value.
   8. A rectangle is constructed, the width of which is the IQR (from LQ to UQ), and a vertical line segment is drawn in the box to represent the median. Horizontal line segments are drawn from the edge of the box to the extremes. Sometimes these are called whiskers.
   9. Tukey formalized the identification of outliers by declaring that any data value that exceeded UQ + 1.5 x IQR or LQ - 1.5 x IQR was an outlier.
   10. When John Tukey was asked why 1.5, he said that 2 was too big and 1 was too small!

C. Now that we have an idea about useful graphical displays and numerical statistics, we will take a look at some real data.
   1. The line plot of the number of medals won at the 1984 winter Olympics, by country, shows that there were two countries whose totals were extremes. If told that those countries were the Soviet Union and East Germany, you could use the display to discuss the importance of sports in Eastern Block countries' attempts to win the Cold War.

2. A histogram of the percentage of votes received by Jimmy Carter, by state, in 1980 shows that he received a majority in only two states. The data are definitely skewed toward the lower end of the distribution.

3. A box plot comparison of the caloric content of beef, meat, and poultry hot dogs reveals that poultry hot dogs, on average, have fewer calories. Although the beef and meat data are skewed to the lower end, the poultry numbers are skewed slightly to the upper end. The plots indicate that the variation in the three distributions is about the same and there are no outliers.

III. EDA techniques can be used to compare sets of data, explore patterns in data, and look for changes in those patterns over time.

A. These techniques can be used to make estimates about the population based on data collected from representative samples.

B. The examples involving Olympic medals, presidential election data, and hot dog calories demonstrate how graphical displays and numerical statistics can be used as tools to analyze data.

C. In Lecture Twenty-One, we will look at the analysis of two-variable data and in Lectures Twenty-Two and Twenty-Three, we will examine quantitative analysis of data.

Essential Reading:

James M. Landwehr and Ann E. Watkins, *Exploring Data.*

David S. Moore and George P. McCabe, *Introduction to the Practice of Statistics,* Chapter 1.

Supplementary Reading:

Sanderson Smith, *Agnesi to Zeno.*

Lynn Arthur Steen, *On the Shoulders of Giants*, UNCERTAINTY.

Questions to Consider:

1. Consider the 1980 data on the percentage of votes received, by state, by Jimmy Carter. In which two states did he receive a majority of the votes? Although the histogram indicated that many of the states had percentages that were less than 40, President Carter did not lose the election by a landslide. Why not?
2. Find a set a data in your newspaper or a newsmagazine. Determine which type of graphical display and which numerical statistics you would use if you were asked to analyze the data.
3. Find a graphical display of data in a newspaper or magazine advertisement. Critically examine the display to see if the graph is constructed in a manner that twists the truth. How would you correct the display?

Lecture Twenty-One

Regression Analysis and Correlation

Scope: In the previous lecture, we investigated the analysis of single-variable data. In this lecture, we examine the analysis of bivariate data, which involves determining a relationship between the two variables. Regression analysis is a method that allows a model to be found that best represents the data. With a proper model, one can make predictions and anticipate problems in advance. Various sets of data and their models will be seen. Correlation is a measure of the relationship between two variables. This index of relationship will be examined and the important difference between correlation and causation will be discussed in depth.

Outline

I. Given a set of paired data, regression techniques allow the analyst to view different types of models. The ultimate goal is to determine the model that best represents the relationship between the two variables.

A. The most common type of model is the linear function. Linear models are simple to understand and use.

1. In algebra, we use $y = mx + b$ for the standard linear equation. In data analysis, various versions of that form of equation are used.
2. Some statistics books use $y = a + bx$, where b is now the slope of the line. This is the formula that we will use.
3. Other sources use $y = b_0 + b_1x$. This form is used in anticipation of multivariate regression analysis where y is predicted by two or more explanatory variables.
4. No matter which form is used, the meanings of the slope and the y-intercept are the same as in algebra.
5. Because the regression equation is found using data from a sample, statisticians consider the coefficients (slope and y-intercept) to be estimates of the true population values. The regression methods are designed to provide coefficients that are valid estimates.

B. In analyzing data comparing the fat content and calories in a slice of pizza, we notice that the data points do not line up. In the real world, there is variation, and the task is to find the linear equation that best "averages" out the relationship between x (fat) and y (calories).

1. The linear model for the pizza data can be found by selecting two representative points and using algebra methods to find the equation of the line containing those two points.

2. Many computers and graphing calculators have regression menus that allow the analyst to obtain the equation of the regression line by pressing a few buttons.
3. Whatever method is used to find the linear model, it must be remembered that this is an estimate. Limits should be placed on the use of the model. Would you want to use the pizza model to predict calories based on fat in dessert pizza?
4. No matter which linear model we choose for our pizza data, the slope is the predicted change in calories if the fat content of a slice of pizza is increased by 1 gram.
5. The y-intercept for the model predicts the amount of calories a slice of pizza would have if there was zero fat content. Calories can be obtained from carbohydrates and protein.

II. Not all regression models are linear. Remember that the rationale for the creation of calculus was the fact that many relationships in the physical world were known to be non-linear.

A. In our lectures on algebra, we discussed the quadratic function. For many sets of data, the second-degree polynomial is a valid model.

1. Looking at the growth of the population density of the United States over time, it is apparent that the data can be modeled with the right half of a parabola ($ax^2 + bx + c$ form).
2. With the exception of three time periods (1800–1810, 1840–1850, and 1950–1960), the population density has increased at an increasing rate.
3. Using a regression menu, we can find the quadratic equation that best relates the year of the census and the population density.
4. This model is useful if we wish to discuss the population growth of the nation or if we wish to make predictions about population density in the near future. It is probably not a valid function of this model to predict population density 60 years in the future.

B. In our excursion in algebra, we discussed exponential functions. This type of function can be used to model data for which the y values are increasing at a rate that is greater than would be predicted by a quadratic model.

1. Federal expenditures on social insurance (Social Security, Medicare, and Medicaid) over the period 1960–1980 appear to be a candidate for an exponential regression model.
2. Using a regression menu, we obtain a valid exponential model. We recall from our discussion of exponential functions that the base (b) indicates the growth rate of the expenditures.
3. We can use this model to predict the expenditures in the year 2050. The acceptance of this estimate must be tempered by the fact that we are using data from the 1960s and 1970s to forecast information about the year 2050.

C. In some cases, the most appropriate model is a piecewise-determined model. A drastic change in the data requires two or more functions to be joined together to represent the data trend.

1. The number of colleges that were members of the NCAA each year provides a set of data that appears to be best modeled by two separate linear equations.
2. This model forces us to ask what happened to so drastically change the slope of the growth of NCAA membership.

D. The cubic model is a useful tool when the data appear to have an inflection point—a location at which the rate of change in the slope changes.

1. One example of a data set that can be represented by a cubic model is the growth in the number of high school soccer players in the United States during the latter part of the twentieth century.
2. A second set of data that appears to be cubic in nature is the life expectancy of Americans during the twentieth century.
3. Both models motivate a concern about what happened to radically change the growth pattern?

E. Some data sets do not require a specific model. A graphical display of the data motivates a discussion of what is revealed and what is predicted.

1. Data on the number of Ph.D. diplomas awarded in mathematics by American universities, by year, and the percentage of those degrees awarded to American citizens present interesting patterns.
2. A mathematical model is not needed to discuss what has been the trend in doctoral degrees in mathematics. Models could be found with the appropriate regression program if a specific function was desired.

III. Once we have found a function to model our bivariate data, the validity of the model must be determined.

A. Typically the criteria used are fit, residual analysis, a justification for the nature of the model, and correlation.

1. The fit can be determined visually. Does the model closely follow the pattern of the data points? Are there points that have an influential effect? Are there points that are not close to the model?
2. Fit can also be measured algebraically. The difference between the actual y value and the predicted y value for a data point is called the residual. If you square the residuals and sum the squares, the sum of squared error is obtained. The best-fit model will minimize the sum of squared error.
3. The earliest work on minimizing the sum of squared error was done by Adrien Legendre in the early part of the nineteenth century.
4. Residual analysis involves analyzing a plot of the residuals versus either x or the y values of the data. Obvious patterns in the graph of

the residuals or patterns in the variation of the residuals could mean that the model being analyzed is not the best model for the data.

5. Some people would use the correlation coefficient as a measure of the validity of a model. This may yield erroneous results.

B. The correlation is an index of the strength of a relationship. The numerical value can vary from -1 through +1.

1. The correlation coefficient can be influenced by a small subset of the data, or even by one point. Data with a correlation coefficient close to -1 or +1 may not be as strongly related as indicated by the correlation value.
2. Correlation should be examined, but it must not be used as a primary indicator of the validity of a model. Fit and residual analysis should be the primary assessment tools.
3. It is common for correlation to be confused with causation. Two variables that have a high degree of correlation may not have a causal relationship.
4. There is a strong positive correlation between the number of Methodist ministers in New England and the amount of rum imported through Boston harbor during the nineteenth and twentieth centuries. This correlation is due to the influence of population growth on both quantities.
5. There is a strong positive correlation between shoe size and size of vocabulary among elementary school students. Does that mean if you stretch your child's feet, his or her vocabulary will grow?
6. There has been a strong negative correlation between the population of Louisiana and the land area of the state. Is the increasing number of people causing the land to erode or is it happenstance?
7. The square of the correlation coefficient is called the "coefficient of determination" and represents the fraction of the change in the response variable (y) that can be explained by its relationship with the explanatory variable (x).

Essential Reading:

David S. Moore and George P. McCabe, *Introduction to the Practice of Statistics*, Chapters 2 and 9.

Supplementary Reading:

John Freund and Ronald Walpole, *Mathematical Statistics*.

James M. Landwehr and Ann E. Watkins, *Exploring Data*.

Questions to Consider:

1. On many occasions, a representative of the tobacco industry has stated that there has not been a scientific experiment that proves that smoking *causes* disease. There have been numerous observational studies that have shown a strong correlation between smoking and disease. What would your response be to the tobacco representative?
2. Obtain data on the growth of the world's population. What type of regression function would you use to model these data? Justify your choice.

Lecture Twenty-Two

The Central Limit Theorem and Hypothesis Testing

Scope: The first two lectures on statistics involve qualitative analysis of data. The next two lectures will use quantitative analysis of data. This is generally called inferential statistics, because we are making an inference about the population based on data from a sample. In this lecture, we will learn about the Central Limit theorem, which is the key to inferential statistics. Using the Central Limit theorem as a base, procedures for testing hypotheses on means and proportions will be thoroughly investigated. The use of the standard normal table, which was developed in an earlier lecture, will be used to assess the level of significance of the inference being made about the hypothesis.

Outline

I. The Central Limit theorem (CLT) provides guidelines for working with sampling distributions. Understanding the meaning of a sampling distribution and being aware of the provisions of the CLT establish a basis for working with hypothesis testing methods.

A. A sampling distribution is the theoretical pattern of the sample statistics of all the possible samples, of equal sample size, drawn from a population.

1. Imagine a large barrel filled with colored marbles. Forty percent of the marbles are red and 60% are blue.
2. After the marbles are thoroughly mixed and you are blindfolded, you reach in and select twenty marbles. An assistant counts the number of red marbles, records the proportion of red marbles in the sample, and returns the twenty marbles to the barrel.
3. The marbles in the barrel are mixed and the process is repeated a nearly infinite number of times.
4. The pattern of the sample proportions, from zero (no red marbles out of twenty) to one (twenty red marbles out of twenty), is the sample distribution.

B. The CLT addresses the mean, standard deviation, and shape of the sampling distribution.

1. Work on the CLT began in 1810 and was accomplished by Pierre Simon Laplace.
2. The CLT requires that the samples are selected randomly and are all the same size.
3. No matter what the shape of the distribution of the population from which the samples were drawn, as the sample size increases, the sampling distribution approaches normality.

4. The expected average value for all the samples is the population parameter. A sampling distribution of sample means has a mean equal to the mean of the population.
5. The expected average for a sampling distribution of sample proportions is the population proportion, or probability.
6. The amount of variation in the sampling distribution is reduced as the sample size increases. Specifically, the theoretical standard deviation of a sampling distribution of means is the population standard deviation divided by the square root of the sample size. The theoretical standard deviation of the sampling distribution of proportions is the square root of pq/n.
7. The theoretical standard deviation is usually referred to as the standard error.

II. Hypothesis testing is a formal process that uses the CLT to make inferences about populations based on information from samples.

A. A hypothesis test begins with the statement of the null and alternate hypotheses. The null hypothesis (H_0) states that there is no significant change between the population represented by the sample and the general population. The alternate hypothesis (H_a) states that there is a significant change between the population represented by the sample and the general population.

1. The null hypothesis is used for the testing process because we know the population parameter that is required by the CLT. The testing process is designed to determine whether the null hypothesis is probably true or probably false.
2. It is assumed that the null hypothesis is true. The sample data are evaluated to see if they contradict the null hypothesis.
3. Because the CLT states that the sampling distribution would be approximately normal (assuming a large enough sample), we use the standard normal table to measure how far away the sample statistic is from the expectation of the null hypothesis.
4. The z score is computed by dividing the difference between the sample statistics and the population parameter (the expectation of the null hypothesis) by the standard error. The standard normal table is used to translate the z score into a probability.
5. The probability of obtaining a sample result such as the actual sample statistic if the null hypothesis is true is the outcome of the process. This probability is called the level of significance or P value.
6. If the level of significance is small enough, there is evidence that the null hypothesis is not true.

B. To perform a hypothesis test on a sample proportion, the sample proportion (p-hat), population probability (p_0), and sample size (n) must be known.

1. The standard error is the square root of pq/n, where q = 1 - p.
2. The z score is computed by z = (p-hat - p_0)/standard error. The level of significance is found in the standard normal table using the z score.
3. A newspaper claims that 40% of the population favors a tax increase. A sample of 100 randomly selected citizens is asked for an opinion and only 30 support the increase.
4. The null hypothesis states that the true proportion is 40%. The alternate hypothesis states that the true proportion is less than 40%.
5. The standard error is calculated to be .049. The score is -2.04 and the level of significance is approximately .02.
6. The sample result seems to indicate that the null hypothesis is not true. The level of significance is quite small. Level of significance relates to the likelihood that the null hypothesis is true.
7. There is also a testing technique for examining the difference between two sample proportions.

C. To perform a hypothesis test on a sample mean; the sample mean (x-bar), population mean (μ_0), the population standard deviation (σ), and sample size (n) must be known.
   1. The standard error is the standard deviation divided by the square root of the sample size.
   2. The z score is computed by z = (x-bar - μ_0)/standard error. The level of significance is found in the standard normal table using the z score.
   3. A manufacturing plant had an average daily production of 30 vehicles per team with a standard deviation of 10. A random sample of 25 teams was allowed to participate in a trial incentive program. These teams had an average daily production figure of 35 vehicles under the incentive plan.
   4. The null hypothesis states that if the entire plant adopted the incentive plan, the average daily production would still be 30 vehicles per team. The alternate hypothesis is that the average production figure with the incentive plan would be greater than 30.
   5. The standard error is calculated to be 2. The z score is 2.50 and the level of significance is approximately .006.
   6. The sample result seems to indicate that the null hypothesis is not true. The level of significance is very small.
   7. There are also hypothesis testing techniques for examining the difference between two sample means and for examining the results of matched pair experiments.

Essential Reading:

David S. Moore and George P. McCabe, *Introduction to the Practice of Statistics*, Chapters 6, 7, and 8.

Supplementary Reading:

John Freund and Ronald Walpole, *Mathematical Statistics*.

Questions to Consider:

1. What is the difference between an experiment and an observational study?
2. Other than to satisfy the CLT, why should samples be randomly selected for hypothesis tests? What are stratification and blocking, and how are they used in hypothesis tests?

Lecture Twenty-Three

Surveys and Confidence Intervals

Scope: You are watching the news on TV and the anchorperson says that a survey of 1000 Americans indicates that 62% of Americans favor a particular bill in Congress with a margin of error of plus or minus 3%. What does this mean? This lecture is designed to answer that question, as well as to explain how the pundits' predictions of the 1948 presidential election were so wrong. At the conclusion of this lecture, you should have a better idea of how a sample of 2000 families provides valid information on the television-watching habits of all Americans.

Outline

I. How do we obtain information about a population? We could ask everyone, but that would be expensive and time consuming. In the case of an election, we could wait until the results are complete and then we would know, but that would not meet the needs of politicians and contributors who want to know what is going to happen in advance.

A. The type of information that is needed about a population is as varied as the individuals and organizations who have the need for that information.

1. A manufacturer wants to know the true percentage of parts that will be defective or the true average life of an AA battery.
2. A financial institution might want to know the true average balance of all retail accounts or what interest rate would inspire savings customers to increase their deposits.
3. A politician might want to know what percent of the state supports a particular initiative or candidate, what percent of the voting population recognizes the politician's name, or what percent of the registered voters definitely plan to vote in the next election.
4. A municipal government might want to know how satisfied parents are with the school system, what the average response time is on a 911 call, or what percentage of the residents of a particular area support a zoning change.

B. In all these situations and many, many more, the information about the population is most obtained through a sampling process (cf., the light bulb example).

1. Sampling can be a very effective means to obtain population information. A census, which gathers data from the entire population, is probably not as accurate as a well-designed sampling process. Many statisticians argue that a sampling should replace the

U.S. census, so that more accurate information would be obtained. The courts have ruled in favor of the census, not because of statistical accuracy, but because of their interpretation of the U.S. Constitution, which mandates a decennial census.

2. The CLT tells us that a sampling distribution is approximately normal if we have a sufficient sample size. We use this knowledge to interpret the sample result into an estimate of the population information that is sought.
3. If we are seeking to estimate a population percentage or probability, we obtain a count and the sample size from our sample. If we are seeking knowledge about a population mean, we obtain the sample mean and sample size from our sample and use the population standard deviation, if known.
4. If the population standard deviation is not known or if the sample size is too small to use the standard normal table, another table called the "Student t table" is used. That table will not be discussed here.

C. Our sample provides us with a point estimate of the population parameter. We know from the CLT that variation among samples prohibits us from claiming that the population parameter is equal to our sample statistic.

1. Using the standard normal table, we calculate an interval centered about the sample statistics. We have confidence that within this interval, the true population parameter will be found.
2. The degree of confidence is expressed as a percentage. Typically, the survey results that are reported in the media have 95% confidence intervals.
3. A 95% confidence interval means that if the sampling that was done was repeated so that there were 1000 samples of the same sample size and if each sample was used to compute a 95% confidence interval, then the true population parameter would be included in 950 of those intervals.
4. We cannot know if our one sample has provided us with an interval that contains the true parameter or if it is one of those 50 samples that do not. Statistics is about uncertainty and that is why this is called inference and not truth.

II. Our knowledge of the standard normal table and the hypothesis test procedures can help us understand confidence intervals.

A. To compute a confidence interval to estimate a population percentage or probability using a sample p-hat, the z score appropriate for the level of confidence must be determined and the standard error must be calculated.

1. If there is 95% confidence, then there is 5% error. Because the error could be to the right or to the left of the interval, we split the

5% into two equal "tails." The z values for 2.5% tails are +/- 1.96. The plus or minus reflects the idea that the interval is above and below the sample statistic.

2. The standard error is similar to the computation used in the hypothesis test. It is the square root of (p-hat times q-hat divided by n), where q-hat is 1 - p-hat.
3. For example, if we randomly selected 100 people and found that 36 of them used public transportation, we could compute a 95% confidence interval to estimate the true percentage of the population that uses public transportation.
4. The sample proportion (p-hat) is .36 and the standard error is .048. The z for the 95% confidence interval is (+/-) 1.96.
5. The 95% confidence interval based on our sample would be .36 +/- .09. That is, we are 95% confident that the true proportion of the population that uses public transport is 36% with a margin of error of 9%.
6. We could also state that we are 95% confident that the true proportion is between 27% and 45%. If we wanted a narrower interval, and hence a smaller margin of error, we would need a larger sample size.

B. We would use similar procedures to compute a confidence interval to estimate a population mean using a sample mean.

1. The standard error, as in the hypothesis test, is the population standard deviation (σ) divided by the square root of the sample size (n).
2. The z score is based on the level of confidence, just as it was for the previous method.
3. A quality-control technician tests 100 randomly selected nine-volt batteries for useful life. The sample average is 78.6 minutes and the standard deviation for nine-volt batteries is known to be 18.3 minutes.
4. The standard error is calculated to be 1.83 and the score must be +/- 1.96 for a 95% confidence interval.
5. The confidence interval is 78.6 +/- 3.6 minutes. We are 95% confident that the true useful life of a battery is 78.6 minutes with a margin of error of plus or minus 3.6 minutes.
6. The interval can be stated as being between 75 minutes and 82.2 minutes.

III. Historically, there have been some very interesting surveys of American presidential elections. There are also surveys that affect our individual lives.

A. Political surveys are conducted all the time, but a few presidential election polls have become classic stories.

1. In 1936, *The Literary Digest* forecast that Alf Landon would easily win the presidential election. This prediction was based on a very

large survey sent to subscribers and those who had automobiles or telephones registered in their names.

2. A young Ph.D. predicted that *The Literary Digest* poll would give erroneous results and he gave his prediction of a landslide for FDR.
3. President Roosevelt won the election in a landslide. *The Literary Digest* eventually stopped publishing. The young unknown Ph.D. was George Gallup.
4. Most pundits, including Gallup, predicted a victory for Thomas E. Dewey in the 1948 election. The demographic information used for selecting the samples in 1948 was based on the 1940 U.S. census.
5. President Truman won and pollsters took a new look at sampling techniques.
6. The 1980 election was considered too close to call, even days before the election. The election turned out to be a significant victory for Ronald Reagan.
7. The large number of people who voted for a third-party candidate and the large number of voters who were not sure of their vote until entering the booth prevented an accurate prediction before the balloting.

B. Surveys, whether we as individuals are part of the sample or not, affect our lives in many ways.

1. TV ratings are p-hats reported as percentages. The cancellation or renewal of shows is based on the numbers obtained from this sampling. Local radio ratings are derived from local samples.
2. Marketing surveys and test markets establish which new products will be available and which products will disappear from the shelves of our stores.
3. Municipal commissions make decisions on planning issues based on surveys.
4. Colleges and community schools will offer courses based on surveys that attempt to gauge local interest.

Essential Reading:

James M. Landwehr, Jim Swift, and Ann E. Watkins, *Exploring Surveys and Information from Samples.*

David S. Moore and George P. McCabe, *Introduction to the Practice of Statistics,* Chapters 6, 7, and 8.

Supplementary Reading:

John Freund and Ronald Walpole, *Mathematical Statistics.*

Questions to Consider:

1. Find a newspaper report of a survey. *The New York Times* is an excellent source. The report should have a feature generally titled "How the Survey Was Conducted." Can you interpret what this feature says given what you now know about confidence intervals? A statement included in the survey description will state that margin of errors for subsets of the sample, such as ethnic groups, will be larger than stated for the entire sample. Why is this true?
2. Research the Old Coke–New Coke marketing tests done in 1984. What was done incorrectly that caused the introduction of the new product to have a negative effect on sales?

Lecture Twenty-Four

A Σumming Up

Scope: This lecture brings the series to a close. By asking the question, "What would our number system look like if our ancestors had only eight (instead of ten) fingers?," we can understand that mathematics is a way of doing things. It is based on rational thinking rather than on arbitrary rules dictated by some elite force. The learning of mathematics can be related to the learning of your native language by learning a new language. For many people, mathematics is a magic language that is incomprehensible. We at The Teaching Company believe that this is not true. There is not a "math gene," nor is the appreciation of mathematics restricted to a special club. We can all appreciate mathematics and use it to improve our lives, both aesthetically and in practical ways. The lecture concludes with a quick review of what the series has presented and one more chance to appreciate the power of mathematics.

Outline

I. What if the earliest human beings had only had eight fingers (including the two thumbs) with which to count? What would be the structure of our number system?

A. We would only have the use of the digits 0, 1, 2, 3, 4, 5, 6, and 7. There would be no eight or nine digit and we would still refer to our number system as base ten.

1. The basic multiplication facts for our system would be:

	1	2	3	4	5	6	7
1	1	2	3	4	5	6	7
2	2	4	6	10	12	14	16
3	3	6	11	14	17	22	25
4	4	10	14	20	24	30	34
5	5	12	17	24	31	36	43
6	6	14	22	30	36	44	52
7	7	16	25	34	43	52	61

2. The largest two-digit number is 77 and the first three-digit number is still 100, but now this represents an amount equivalent to 64 (8 x 8) in our normal system.

3. All of our knowledge about arithmetic and algebra would be based on these numbers and we would laugh at the concept of a number system with two more digits.

4. By analyzing this multiplication system, many patterns can be revealed to us. In many cases, these patterns exist in our own system but we are too familiar with the system to look for patterns.

B. What do we learn from inspecting a number system that uses a different base?

1. To me, examining a number system with a different base is similar to taking French in school to enhance one's knowledge of English.
2. I did not know what a participle was until I took French in seventh grade. I obviously used participles and inherently understood rules about their use, but I had never thought about their meaning.
3. Learning the rules of another language allowed me to investigate my own language and learn things about the language that had never entered my mind.
4. Imagine if we had ancestors with twelve fingers. We would find a base-ten number system confusing at first, but with patience we would adapt ourselves to the new system.
5. It is the same thing with mathematics in general. Patience and perseverance will be rewarded with knowledge and confidence.

C. I hope this tour of the beauty, the functions, and the history of mathematics has allowed those who had fears related to mathematics to reduce their anxiety.

1. Mathematics is not for a restricted elite. Failure to understand and appreciate mathematics can lead to a lessening of the quality of one's life.
2. Some people believe that knowledge of mathematics is similar to the knowledge of Latin during the Middle Ages. Only a select few were allowed to learn Latin. Most available knowledge was written only in Latin and the mass of people was restricted from exposure to new knowledge.
3. In a world where technology is expanding more rapidly than ever before in human history, knowledge of mathematics is available and is vital for every person, no matter what his or her vocation or avocation.

D. Let us quickly review the significant parts of this series.

1. Arithmetic is the result of thousands of years of human development. Its rules are not arbitrary but are based on both logical and consistent foundations.
2. Algebra is a generalization of arithmetic and is a powerful tool in describing our environment and the consequences of human action.
3. Calculus applies algebraic knowledge to measure change and the cumulative effects of change.
4. Our brief look at fractals should entice you to delve into this modern branch of mathematics. Even if one is confused by the abstract mathematics involved in creating fractal images, those images offer the viewer amazing beauty and mysteries that have yet to be deciphered.
5. Geometry may be the most concrete branch of mathematics and it is something that we use every day, usually without thinking about what knowledge we are using.
6. Probability allows us to measure and manipulate uncertainty. This is critical in a world in which uncertainty appears to be growing. The use of simulations gives us tools to answer questions that, in the past, might have been without answer.
7. Our discussion of data analysis should have made you familiar with the collection, display, and analysis (both qualitative and quantitative) of data. The word "statistics" is sometimes replaced with "sadistics" by people who have suffered through a college course with dozens of formulas and rules that did not always provide an answer. Statistics is not a cruel science, but rather a useful tool for interpreting the present and forecasting the future.

II. Let us close our series with a review of the power of mathematics.

A. Mathematics is a powerful universal language and a tool that is required if we are to solve problems.

1. It allows us to represent complex ideas with simple symbols.
2. Models can be constructed to attempt to both explain and to predict.
3. Simulations can be used to enhance our ability to define problems and test solutions.
4. Mathematics can be used to classify objects and concepts and to translate and transport ideas.

B. Mathematics is the language of science.

1. Applications of mathematics in the science of genetics allow us to learn about the transfer of characteristics from one generation to the next.
2. The balancing of chemical equations provides models for the creation of substances to solve problems.
3. Through the use of symbolic equations and systems of equations, we can model and attempt to control our physical world.
4. The process of digitizing allows rapid communication of scientific ideas.
5. Many of the technological advances in medicine would be impossible without the availability of the appropriate mathematics.

C. Mathematics is a source of pleasure.

1. Probability-based games provide entertainment.
2. Geometric principles give form and reality or abstraction to art.
3. Mathematical knowledge allowed the ancients to produce harmonious sounds for aural pleasure. That has not changed.

Technology has simply given us more ways to use mathematics to create music.

4. Even our negative thoughts about mathematics can be sources of enjoyment through humor. A famous cartoon shows a gentleman arriving at the pearly gates. He is told that his admission to heaven will be based on the answer to one question. That question is: "A train leaves Chicago. At the same time another train leaves Los Angeles…" He'd better hope he studied his algebra!

Essential Reading:

John Allen Poulos, *A Mathematician Reads the Newspaper* and *Innumeracy*.

Supplementary Reading:

Theoni Pappas, *The Magic of Mathematics*.

Questions to Consider:

1. Some time ago there was a flurry of complaints about a Barbie doll that proclaimed, "Math is tough." How would you answer Barbie?
2. Make a list of your activities in a single day that use mathematics. Thinking back to your formal education process, what should have been done differently in that process to better prepare you to use mathematics in your daily life?

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z-table

Z	0	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
-3.4	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0002
-3.3	0.0005	0.0005	0.0005	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0003
-3.2	0.0007	0.0007	0.0006	0.0006	0.0006	0.0006	0.0006	0.0005	0.0005	0.0005
-3.1	0.0010	0.0009	0.0009	0.0009	0.0008	0.0008	0.0008	0.0008	0.0007	0.0007
-3.0	0.0013	0.0013	0.0013	0.0012	0.0012	0.0011	0.0011	0.0011	0.0010	0.0010
-2.9	0.0019	0.0018	0.0018	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
-2.8	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
-2.7	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026
-2.6	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	0.0038	0.0037	0.0036
-2.5	0.0062	0.0060	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
-2.4	0.0082	0.0080	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
-2.3	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
-2.2	0.0139	0.0136	0.0132	0.0129	0.0125	0.0122	0.0119	0.0116	0.0113	0.0110
-2.1	0.0179	0.0174	0.0170	0.0166	0.0162	0.0158	0.0154	0.0150	0.0146	0.0143
-2.0	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
-1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.0250	0.0244	0.0239	0.0233
-1.8	0.0359	0.0351	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0301	0.0294
-1.7	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
-1.6	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
-1.5	0.0668	0.0655	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0571	0.0559
-1.4	0.0808	0.0793	0.0778	0.0764	0.0749	0.0735	0.0721	0.0708	0.0694	0.0681
-1.3	0.0968	0.0951	0.0934	0.0918	0.0901	0.0885	0.0869	0.0853	0.0838	0.0823
-1.2	0.1151	0.1131	0.1112	0.1093	0.1075	0.1056	0.1038	0.1020	0.1003	0.0985
-1.1	0.1357	0.1335	0.1314	0.1292	0.1271	0.1251	0.1230	0.1210	0.1190	0.1170
-1.0	0.1587	0.1562	0.1539	0.1515	0.1492	0.1469	0.1446	0.1423	0.1401	0.1379
-0.9	0.1841	0.1814	0.1788	0.1762	0.1736	0.1711	0.1685	0.1660	0.1635	0.1611
-0.8	0.2119	0.2090	0.2061	0.2033	0.2005	0.1977	0.1949	0.1922	0.1894	0.1867
-0.7	0.2420	0.2389	0.2358	0.2327	0.2296	0.2266	0.2236	0.2206	0.2177	0.2148
-0.6	0.2643	0.2709	0.2676	0.2643	0.2611	0.2578	0.2546	0.2514	0.2483	0.2451
-0.5	0.3085	0.3050	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.2810	0.2776
-0.4	0.3446	0.3409	0.3372	0.3336	0.3300	0.3264	0.3228	0.3192	0.3150	0.3121
-0.3	0.3821	0.3783	0.3745	0.3707	0.3669	0.3632	0.3594	0.3557	0.3520	0.3483
-0.2	0.4207	0.4168	0.4129	0.4090	0.4052	0.4013	0.3974	0.3936	0.3897	0.3859
-0.1	0.4602	0.4562	0.4522	0.4483	0.4443	0.4404	0.4364	0.4325	0.4286	0.4247
0.0	0.5000	0.4960	0.4920	0.4880	0.4840	0.4801	0.4761	0.4721	0.4681	0.4641
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5329	0.5279	0.5319	0.5319
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141

z-table

Z	0	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7910	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998